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BIOGRAPHY.

JOHN HENRY LAMBERT.

BY DR. GEORGE BRUCE HALSTED.

JOHN HENRY LAMBERT was born August 29, 1728, at Muehlhausen in Alsace. Through unaided study he mastered the then ordinary mathematics. In 1745 he became secretary to Professor Iselin editor of a newspaper at Basel, who three years later obtained his appointment as private tutor to the family of the Swiss President A. von Salis of Coire.

After completing with his pupils a tour of two years duration through Goettingen, Utrecht, Paris, Marseilles, and Turin, he resigned his position in 1759 and settled at Augsburg. Afterward Munich, Erlangen, Coire, Leipzig became for brief successive intervals his home.

Finally in 1764 he settled in Berlin, and was elected a member of the Royal Academy of Sciences.

He died of consumption on September 25th, 1777. Among his most remarkable works are *Pyrometrie* (Berlin, 1779) a systematic treatise on heat; *Photometria* (Augsburg, 1760); *Insigniores orbitae Cometarum proprietates* (Augsburg, 1761). Beitrage zum Gebrauche der Mathematis und deren Anwendung (4 Vols. Berlin, 1765-72).

It is to Lambert, another Leibnitz as to the variety and depth of his attainments and views, that is due the law of the conduction of heat in a small metallic bar exposed at one end to the constant action of a source of heat.

Experiment confirmed his theoretic results, which become the basis of further researches and of the tremendously important Theory of heat of the mathematician Fourier.

Though belonging to a time when the wonder-working Calculus lus and analytic mathematics dazzled all, yet Lambert, though the associate at

John Henry Lambert.

Berlin of Lagrange and Euler, nevertheless retained the taste for pure geometry and understood how to make of it the most surprising applications. He demonstrated synthetically his celebrated theorem on the arcs of ellipses of the same major axis, described in the same time. His *Freye Perspective*, published in 1759, and increased by a second part in 1773, makes use of this theory as of a geometric method and proves many theorems relating to the descriptive properties of figures. He is the true forerunner of Monge. Again, the problem of the arithmetical quadrature of the circle is as old as mathematics.

Lambert it was who first proved the task of the π-computers endless by demonstrating that π is irrational. This alone would have made him immortal. The proof communicated by Lambert to the Berlin Academy of Sciences in 1761 is reproduced in Note IV. of Legendre's *Geometrie.*

Lambert developed De Moivre's theorems on the trigonometry of complex variables and introduced the hyperbolic functions which he designated by *sinh x*, *cosh x*, &c.

His work must therefore have attracted the especial attention of Gauss, one of whose chief claims is the development of the theory of complex variables.

In still another direction Lambert deserved immortality. He was the originator of Symbolic Logic.

[See my article *Symbolic Logic* in Johnston's New Universal Encyclopaedia]. He fully recognized that the four algebraic operations, addition, subtraction, multiplication, division, have each an analogue in logic, namely *Zusammensetzung*, *Absonderung*, *Bestimmung*, *Abstraction*, which may be symbolized by $+$, $-$, $\times$, $\div$. He also preceived the *inverse* nature of the second and fourth as compared with the first and third. He enunciates with perfect clearness the principal logic laws, such as the commutative and distributive. He *develops* simple logical expressions precisely as Boole did later. He interpreted and represented hypothetical propositions precisely as Boole did. In one passage at least he recognized that the inverse process, marked by division, is an *indeterminate* one.

Venn says: "To my thinking he and Boole stand quite supreme in this subject in the way of originality." Even Kant calls Lambert "*der unvergleichliche Mann.*"

In the very short and imperfect sketch of Lambert by F. W. Cornish of Eton College inserted in the Encyclopaedia Britannica in 1882 we read: "In Bernouilli and Hindenburg's *Magazin* (1787-1788) he treats of the roots of equations and *of parallel lines.*" From this lover of pure geometry, this prophetic pioneer in logic, in mathematic, in their combination, that could only mean the non-Euclidean geometry. And so it was.

The essay, "Zur Theorie der Parallellinien," was written in September, 1766, but first published in 1786 by F. Bernouilli (a kinsman of John Bernouilli) from the papers left by Lambert, and appears in the *Leipziger Magazin fuer reine und angewandte Mathematik*, herausgegeben von J. Bernoulli und C. F. Hindenburg, erster Jahrgang, 1786, Seite 137 ff.

In this remarkable work Lambert maintains:

(1) The Parallel-Axiom needs a proof, since it does not hold for the geometry on a sphere.

(2) In order to bring before the perceptive intuition a geometry in which the triangle's angle-sum is less than the two right angles we need the help of an "imaginary sphere."

(3) In a space in which the triangle's angle-sum is different from two right angles, there is an absolute measure [a natural unit for length].

Just as Lambert's celebrated memoir, "Vorlaeufige Kentnisse fuer die, so die Quadratur und Rectification des Circuls suchen," closed a question whose history comprises four thousand years, and which therefore pertains to the very oldest problems of mankind, so this essay of Lambert "Zur Theorie der Parallellinien" should have ended the equally fruitless and almost equally ancient striving after a proof of the Postulatum of Euclid, the ordinary assumption for the treatment of parallels, or anything equivalent to it. Will one man ever again wound to the death two such dragons feeding for centuries on human brains!

In a review of my translation of Vasiliev's Address on Lobachevski Mr. Charles S. Pierce says in *The Nation*, April 4th, 1895; "However, Gauss was not the first discoverer. Lambert in 1785, in a printed book, spoke plainly of a space where the angles of a triangle should sum up to less than 180 degrees, and mentions one of its most remarkable properties. Guass most likely knew of this."

Should this be so, the last claim left to be made for Guass in the determined and persistent endeavor of his German admirers to keep him prominently figuring in the history of the greatest achievment of modern culture, the non-Euclidean geometry, namely the claim that he was the first to recognize with complete clearness the uselessness of all attempts to prove the eleventh axiom, becomes meaningless.

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Professor of Mathematics, University of Michigan, Ann Arbor, Michigan.

(Continued from the June Number.)

In what precedes we have endeavored to convey some of the most important general concepts in regard to the theory of substitution groups, we proceed to some more special concepts leading up to the branch of this subject which we desire to pursue first, viz. the *construction* of the substitution groups containing a given number of letters.

We shall not aim at a rigorous development of any part of the subject but rather at conveying some of the most fruitful concepts and thus to aid (1) those who desire to form a good general idea of the subject without entering

into details and (2) those who desire to acquaint themselves with such facts and methods as may be helpful in a more thorough study. The methods used will in general, be such as lead to a rigorous development, but, for the sake of simplicity we shall not pursue them sufficiently far to call our treatment a rigorous development. "In presenting any mathematical theory it is in general, not so important that a rigerous presentation of the subject is given throughout as that the methods used are those which serve as a means for a rigorous presentation.*

The theory of groups is divided into two parts, viz. *the theory of substitution groups* and *the theory of transformation groups.* To these we might perhaps add *the theory of abstract groups*, which was first suggested by Cayley. Its foundation was laid by Dyck in his noted articles in the Mathematische Annalen for 1882 and 1883. This theory has not yet been extensively developed. The theory of transformation groups was founded by Sophrus Lie who began to publish his results in 1870. It bears the same relations to the solution of differential equations as the theory of substitution groups bears to the solution of the algebraic equations. It has not yet had time to take a definite station among the other branches of mathematics but the indications are that it will occupy a very high one.

The theory of substitution groups is considerably older than the two just named. It might be said to have been founded by Abel† but traces of it are found much earlier, as for instance in the writings of Lagrange.§ The most prominent among those who have contributed to its development are Cauchy, Galois and Jordan. Jordan's "Traite des Substitutions et des Equations" is the standard treatise. After this come Netto's "Substitutionentheorie und ihre Anwendungen auf die Algebra" and the chapters on this subject in Serret's "Cours d'Algebre Superieure," second volume. The only work on the subject in the English language is Professor Cole's translation of Netto's work. A great part of the knowledge of this, as well as of most other modern mathematical subjects, is to be found only in the journals.

Having found the place which the theory of substitution groups occupies in the general theory of groups we proceed to locate the theory of the construction of groups in the more general theory of substitution groups. Since groups are merely instruments of operation it follows that the study of the construction of groups must be the study of instruments and not the more useful study of their uses and the methods of using them. It is evident that the study of the use and the modes of using an instrument can, as a rule, be prfitably pursued only after a thorough acquaintance had been formed with the instrument itself and it therefore seems proper for us first to pay attention to its study.

Another advantage in beginning in this way is the fact that a thorough

* Weberhaupt dürfte es ja bei der Darlegung einer mathematischen Theorie wenger auf eine durchweg strenge Darstellung, als villmehr darauf ankommen, dass die angegebenen methoden die zur strengen Darstellung erforderlichen mittal gewahren. Neumann, S. VIII Riemann's Theorie der Abel'schen Integrale.

†Crelle's Journal für Mathematik, vol. I. 1826

§Dr. James Pierpont: Bulletin of the American Mathematical Society, May, 1895.

acquaintance with a complex instrument is apt to suggest uses and modes of using which we do not find fully described by others. In fact, substitution groups constitute such a complex instrument that it seems almost impossible to gain a thorough knowledge of it from descriptions alone, especially since descriptions, as a rule, are unattractive to those who do not already possess kindred concepts which require only slight modification.

We have already remarked that the process of finding one substitution which is equivalent to two successive substitutions is called multiplication and that the commutative law does not hold in this multiplication. On this account it is necessary to distinguish between multiplier and multiplicand by the order in which they are written. On this point there is no uniformity among the writers on this subject but we shall always suppose that the multiplier precedes the multiplicand. Thus we have

$$abc\ .\ ab=bc.$$

For in the first substitution a is replaced by b and in the second this b is replaced by a, thus a remains unchanged. In the first substitution b is replaced by c and in the second substitution this c is unchanged, hence the two successive substitutions replace b by c. In the first substitution c is replaced by a and in the second this a is replaced by b, hence the two successive substitutions replace c by b. Take for example the expression

$$a+3b+2c.$$

After applying the first substitution this becomes

$$b+3c+2a.$$

If we now operate with the second substitution this becomes

$$a+3c+2b.$$

which is the same result as we should have obtained by operating upon the first expression by bc.

In constructing groups we shall have to use multiplication to a very large extent. The process, is however, very simple as may be inferred from the preceding example and others which have been given before. The great importance of this operation led us to introduce this additional example with explanations. Referring to the lists of groups of two, three, and four letters, we desire to call attention to an important property. Let us consider, for instance the two groups $1,\ ab\ .\ cd,\ ac\ .\ bd,\ ad\ .\ bc$

and $1,\ ac,\ bd,\ ac\ .\ bd$

In the first one we observe that each letter is replaced by every other letter; while in the second a is replaced by c but not by either b or d, similarly d is replaced by b but not by a or c. Groups, like the first, in which one letter is replaced by every other letter of the group are called *transitive* groups. Those, ike the second, in which no letter is replaced by all of the others are called *intransitive* groups. In the preceding lists there are only two intransitive groups but when the number of letters exceeds five this class of groups is by far the larger of the two.

The methods used to construct the intransitive groups are much simpler than those used to construct the transitive, on this account we shall consider the construction of the intransitive groups first.

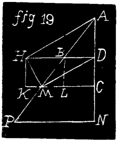
[To be continued.]

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton) Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the June Number.]

PROPOSITION XX. *Let there be a triangle ACM (fig. 19.) right-angled at C. Then from the point B bisecting this AM let fall the perpendicular BD to AC. I say this perpendicular will not be (in the hypothesis of acute angle) greater than half the perpendicular MC.*



Proof. For let DB be produced to DH double DB. Therefore DH would be (if DB be greater than the aforesaid half) greater than CM, and therefore equal to a certain continuation CMK.

Join AH, HK, HM, MD.

Now we proceed thus. Since in the triangles HBA, DBM, the sides HB, BA are assumed equal to the sides DB, BM; and (Eu. I. 15) the angles at the point B are equal; also (Eu. I. 4) the base HA will be equal to the base MD.

Then, by the same reasoning, in the triangles HBM, DBA, the bases HM, DA will be equal.

Wherefore in the triangles MHA, ADM, (Eu. I. 8) the angles MHA ADM will be equal. Again in the triangles AHB, MDB, the residual angle MHB will remain equal to the residual right angle ADB. Therefore the angle MHB will be right. But this is absurd in the hypothesis of acute angle; since the straight KH joining equal perpendiculars KC, HD, makes (P. III.) acute angles with these perpendiculars.

Therefore the perpendicular BD is not (in the hypothesis of acute angle) greater than the half of the perpendicular MC. Quod erat demonstrandum.

PROPOSITION XXI. *The same remaining; if AM, and AC are understood as produced in infinitum. I say their distance (in either hypothesis, of right angle or of acute angle) will be greater than any assignable finite length.*

Proof. In AM produced assume AP double AM, and let fall to AC produced the perpendicular PN.

The perpendicular MC will not be in either of the aforesaid hypothesis greater than half the perpendicular PN.

Therefore PN will be at least double MC, just as MC is at least double BD. And so always, if in AM produced is assumed double AP, and from the termination of this a perpendicular is let fall to AC produced.

It is obvious the perpendicular, which from AM ever more produced is let fall to AC produced, will be a multiple of the determinate BD beyond any finite assignable number.

Therefore the distance of the aforesaid straights will be (in either aforesaid hypotehsis) greater than any assignable finite length.

Quod erat demonstrandum.

THE INTRINSIC EQUATION OF A CURVE IN POLAR CO-ORDINATES.

By G. B. M. ZERR, A. M., Professor of Sciences, Texarkana, Texas.

The intrinsic equation of a curve has never been treated in polar co-ordinates in works on the subject. It is the aim of this paper to treat this subject briefly and to show that this method is somewhat simpler than the method of rectangular co-ordinates.

Let S denote the length of an arc of the curve $r=f(\theta)$ measured from some fixed point, φ the inclination of the tangent at the variable extremity to the tangent at the fixed point.

Let $y=f(x)$, then $\dfrac{dy}{dx}=f'(x)=-\cot\varphi \quad x=r\cos\theta, \quad y=r\sin\theta$

$$\frac{dx}{d\theta}=\cos\theta\frac{dr}{d\theta}-r\sin\theta,\ \frac{dy}{d\theta}=\sin\theta\frac{dr}{d\theta}+r\cos\theta.$$

$$\frac{dy}{dx}=\frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}}=\frac{\sin\theta\dfrac{dr}{d\theta}+r\cos\theta}{\cos\theta\dfrac{dr}{d\theta}-r\sin\theta}=-\cos\varphi$$

$$\therefore\ r\sin\theta\cot\varphi-\cos\theta\cot\varphi\frac{dr}{d\theta}=\sin\theta\frac{dr}{d\theta}+r\cos\theta.$$

$$\therefore\ \frac{dr}{d\varphi}=r\tan(\theta-\varphi)=f'(\theta)$$

$\therefore f'(\theta)=f(\theta)\tan(\theta-\varphi)\ldots.A$-

From A, θ is known in terms of φ, san $F(\varphi)$.

Then $\dfrac{d\theta}{d\varphi}=F'(\varphi)$.

$$\text{Now } \frac{ds}{d\theta}=\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}=\left[\left\{f(\theta)\right\}^2+\left\{f'(\theta)\right\}^2\right]^{\frac{1}{2}}$$

$$\therefore\ \frac{ds}{d\theta}=\left\{r^2+r^2\tan^2(\theta-\varphi)\right\}^{\frac{1}{2}}=r\sec(\theta-\varphi)=f(\theta)\sec(\theta-\varphi).$$

$$\frac{ds}{d\varphi}=\frac{ds}{d\theta}\cdot\frac{d\theta}{d\varphi}=F'(\varphi)f(\theta)\sec(\theta-\varphi)=rF'(\varphi)\sec(\theta-\varphi)\ldots.(B).$$

From B by substituting the value of r in terms of θ, and the value of θ, as found in A, in terms of φ, S may be found in terms of φ by integration.

I. Required the intrinsic equation of the circle.

The polar equation is $r=a$.

$$\therefore\ \frac{dr}{d\theta}=0=r\tan(\theta-\varphi).\quad \therefore\ \theta-\varphi=0,\text{ or }\theta=\varphi\ldots.(1).$$

$$\frac{d\theta}{d\varphi}=1,\ \therefore\ \frac{ds}{d\varphi}=a\sec(\varphi-\varphi)=a\sec 0=a.\quad \therefore\ S=a\varphi.$$

Or we may have proceeded thus:

$$\frac{ds}{d\theta}=\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}=a, \quad \therefore\ S=a\theta, \text{ but } \theta=\varphi, \quad \therefore\ S=a\varphi.$$

Taking the equation $r=2a\cos\theta\ \frac{dr}{d\theta}=-2a\sin\theta=r\tan(\theta=\varphi)$.

$$\therefore -2a\sin\theta=2a\cos\theta\tan(\theta-\varphi).$$

$$\therefore -\tan\theta=\tan(\theta-\varphi), \ \therefore\ \theta=\frac{\varphi}{2}\ldots.(2).$$

$$\frac{d\theta}{d\varphi}=\tfrac{1}{2}. \quad \frac{ds}{d\varphi}=\tfrac{1}{2}\cdot 2a\cos\theta\sec(\theta-\varphi)=a\cos\frac{\varphi}{2}\sec\frac{\varphi}{2}=a,$$

$$\frac{ds}{d\varphi}=a,\ S=a\varphi, \quad \text{or } \frac{ds}{d\theta}=\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}=2a,\ S=2a\theta, \text{but } \theta=\frac{\varphi}{2},\ \therefore\ S=a\varphi.$$

II. Required the intrinsic equation of the parabola. The polar equation is $r=\dfrac{2a}{1+\cos\theta}=\dfrac{a}{\cos^2\frac{\theta}{2}}$.

$$\frac{dr}{d\theta}=\frac{2a\sin\theta}{(1+\cos\theta)^2}=\frac{a\sin\frac{\theta}{2}}{\cos^3\frac{\theta}{2}}=r\tan(\theta-\varphi)=\frac{a\tan(\theta-\varphi)}{\cos^2\frac{\theta}{2}}.$$

$$\therefore \tan\frac{\theta}{2}=\tan(\theta-\varphi), \quad \therefore\ \theta=2\varphi\ldots.(3).$$

$$\frac{d\theta}{d\varphi}=2. \quad \frac{ds}{d\varphi}=\frac{2a}{\cos^2\frac{\theta}{2}}\sec(\theta-\varphi)=\frac{2a}{\cos^3\varphi}.$$

$$\therefore\ S=\frac{a}{2}\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)+\frac{a\sin\varphi}{1-\sin^2\varphi} \quad \text{or thus}$$

$$\frac{ds}{d\theta}=\frac{a}{\cos^3\frac{\theta}{2}}. \quad \therefore\ S=\frac{a}{2}\log\left(\frac{1+\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}}\right)+\frac{a\sin\frac{\theta}{2}}{1-\sin^2\frac{\theta}{2}}, \text{ but } \theta=2\varphi.$$

$$\therefore\ S=\frac{a}{2}\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)+\frac{a\sin\varphi}{1-\sin^2\varphi}.$$

III. Find the intrinsic equation to the Cardioid. The polar equation is $r(1+\cos\theta)=2a\cos^2\frac{\theta}{2}$. $\frac{dr}{d\theta}=-2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$$=r\tan(\theta-\varphi)=2a\cos^2\frac{\theta}{2}\tan(\theta-\varphi).$$

$$\therefore -\tan\frac{\theta}{2}=\tan(\theta-\varphi), \ \therefore\ 2\varphi=3\theta,\ \frac{ds}{d\theta}=\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}$$

$$=2a\cos\frac{\theta}{2}, \quad \therefore\ S=4a\sin\frac{\theta}{2}=4a\sin\frac{\varphi}{3}.$$

IV. Find the intrinsic equation to the curve

$$r^{\frac{1}{m}}=a^{\frac{1}{m}}\cos\frac{\theta}{m} \text{ or } r=a\left(\cos\frac{\theta}{m}\right)^{m}. \quad \frac{dr}{d\theta}=-a\sin\frac{\theta}{m}\left(\cos\frac{\theta}{m}\right)^{m-1}$$

$$=r\tan(\theta-\varphi)=a\left(\cos\frac{\theta}{m}\right)^{m}\tan(\theta-\varphi).$$

$$\therefore -\tan\frac{\theta}{m}=\tan(\theta-\varphi). \quad \therefore\ (m+1)\theta=m\varphi. \quad \frac{ds}{d\theta}=\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}$$

$$=a\left\{\left(\cos\frac{\theta}{m}\right)^{2m}+\left(\sin\frac{\theta}{m}\right)^{2}\left(\cos\frac{\theta}{m}\right)^{2m-2}\right\}^{\frac{1}{2}}=a\left(\cos\frac{\theta}{m}\right)^{m-1}$$

$$S=a\int\left(\cos\frac{\theta}{m}\right)^{m-1}d\theta=\frac{ma}{m+1}\int\left(\cos\frac{\varphi}{m+1}\right)^{m-1}d\varphi.$$

Let $n=\frac{1}{m}$, then $S=\frac{a}{n+1}\int\left(\cos\frac{n\theta}{n+1}\right)^{\frac{1}{n}-1}d\varphi$ is the intrinsic equation to the curve $r^n=a^n\cos n\theta$.

V. Find the intrinsic equation to the Logarithmic Spiral. The polar equation is $r=be^{\frac{\theta}{c}}$ or $r=ba^{\theta}$, $\log\frac{r}{b}=\frac{\theta}{c}$, $\frac{dr}{d\theta}=\frac{r}{c}=r\tan(\theta-\varphi)$.

$$\therefore\ \tan(\theta-\varphi)=\frac{1}{c}, \quad \theta=\tan^{-1}\frac{1}{c}+\varphi=d+\varphi.$$

$$\frac{ds}{d\theta}=\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}=\left(\frac{r^2d\theta^2+dr^2}{d\theta}\right)^{\frac{1}{2}}=\frac{\sqrt{1+c^2}\,dr}{d\theta}.$$

$$\therefore\ ds=\sqrt{1+c^2}\,dr, \quad S=\sqrt{1+c^2}\ .\ r=\sqrt{1+c^2}\,be^{\frac{\theta}{c}}=\sqrt{1+c^2}\,be^{\frac{d+\phi}{c}}.$$

The intrinsic equation for the evolute and involute can be found in the usual way.

A QUADRATIC CREMONA TRANSFORMATION DEFINED BY A CONIC.

By LEONARD E. DICKSON, M. A., Fellow in The University of Chicago.

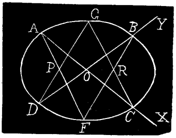
On an arbitrary conic choose four fixed points A, B, C, D. To each point P of the plane there corresponds a definite point R defined by the following construction: Join PA and PD cutting the conic again in F and G respectively. Then R is the intersection of BF with CG.

Let the equation of the base conic referred to the axes AC and BD have the general form $Ax^2+2Hxy+By^2+2Gx+2Fy+C=0\ldots.(1)$.

We may express the equation of the conic in terms of its four intercepts a, b, c, d on the axes and an unknown parameter. Since the points $A(-a,\ 0)$, $B(0,\ b)$, $C(c,\ 0)$, $D(0,\ -d)$ lie on the conic, we obtain the relations:

$$Aa^2-2Ga+C=0$$
$$Bb^2+2Fb+C=0$$
$$Ac^2+2Gc+C=0$$
$$Bd^2-2Fd+C=0.$$

From the first and third, $2G=A(a-c)$; from the second and fourth, $2F=-B(b-d)$. Hence $C=-Aac=-Bbd$. Substituting in



(1); dividing by A, and writing $h=\dfrac{2H}{A}$,

$bdx^2+hbdxy+acy^2+bd(a-c)x+ac(d-b)y-abcd=0\ldots.(2)$. The discriminant of (2), $\dfrac{-ac}{4b^2d^2}\{ab^2c+bc^2d+cd^2a+da^2b+bdh(a-c)(b-d)-h^2b^2d^2\}$, will be 0 only when $h=\dfrac{ab+cd}{bd}$ or $h=\dfrac{-(ad+bc)}{bd}$.

In the former case, the conic (2) becomes $(bx+cy-bc)(dx+ay+ad)=0$, representing the straight lines AD and BC. We reject this trivial case, since to every point in the plane there corresponds the line BC. The quantity $ab+cd-hbd$ occurs below repeatedly.

For the case $h=\dfrac{-(ad+bc)}{bd}$, (2) becomes $(bx-ay+ab)(dx-cy-cd)=0$, representing the straight lines AB and DC. This case is not in the least trivial.

To find the co-ordinates x_1, y_1 *of the point* R, *corresponding to a point* $P(x',\ y')$.

The equation to AP is $y'(x+a)=y(x'+a)$; that to DP is $x(y'+d)=x'(y+d)$. The co-ordinates of the second point of intersection F of AP with the conic (2) are:

$$\frac{c\{bd(x'+a)^2+ay'(b-d)(x'+a)-a^2y'^2\}}{bd(x'+a)^2+hbdy'(x'+a)+acy'^2},$$

$$\frac{y'\{ay'(bc-cd+hbd)+bd(a+c)(x'+a)\}}{bd(x'+a)^2+hbdy'(x'+a)+acy'^2}$$

The co-ordinates of G, the second intersection of DP with the conic (2) are:

$$\frac{x'\{x'bd(c-a+dh)+ac(b+d)(y'+d)\}}{bdx'^2+bdhx'(y'+d)+ac(y'+d)^2},$$

$$\frac{-b\{d^2x'^2+dx'(a-c)(y'+d)+ac(y'+d)^2\}}{bdx'^2+bdhx'(y'+d)+ac(y'+d)^2}.$$

The equation to BF is

$$\frac{x}{y-b}=\frac{-c(ay'+dx'+ad)(ay'-bx'-ab)}{d\{bx'+(hb-c)y'+ab\}\{ay'-bx'-ab\}}=\frac{-c(ay'+dx'+ad)}{d\{bx'+(hb-c)y'+ab\}},$$

say $=\frac{A}{B}$.

The equation to CG is

$$\frac{x-c}{y}=\frac{\{dx'-cy'-cd\}\{b(a-dh)x'-acy'-acd\}}{b(dx'-cy'-cd)(dx'+ay'+ad)}$$

$$=\frac{b(a-dh)x'-acy'-acd}{b(dx'+ay'+ad)}, \quad \text{say}=\frac{D}{E}.$$

By elimination, the co-ordinates of R are:

$$x_1=\frac{A(bD+cE)}{AE-BD}, \quad y_1=\frac{E(bA+cB)}{AE-BD}.$$

Now $AE-BD$ expanded gives $-(ab+cd-bdh)(bdx'^2+acy'^2+bdhx'y'+abdx'+acdy')$; $bD+cE=bx'(ab+cd-bdh)$; $bA+cB=-cy'(ab+cd-bdh)$.

$$\therefore x_1=\frac{bcx'(dx'+ay'+ad)}{bdx'^2+acy'^2+bdhx'y'+abdx'+acdy'},$$

$$y_1=\frac{bcy'(dx'+ay'+ad)}{bdx'^2+acy'^2+\ldots\ldots}\ldots.(3).$$

Hence $\frac{x_1}{y_1}=\frac{x'}{y'}\ldots..(4)$, or PR always passes through O, Pascal's Theorem for a hexagon inscribed in a conic.

Solving (3) for x' and y',

$$x'=\frac{adx_1(-bx_1-cy_1+bc}{bdx_1^2+acy_1^2+bdhx_1y_1-abcy_1-bcdx_1}, \quad y'=\frac{ady_1(-bx_1-cy_1+bc)}{bdx_1^2+\ldots.}\ldots.(5).$$

The reciprocity between P and R shown by (3) and (5) is evident geometrically.

If P describes a straight line $y=mx+l$, the locus of R is a conic.

Substituting the value (5) in $y'=mx'+l$, and dropping the subscripts to the co-ordinates of R, we find its locus: $bdx^2(l-ma)+dxy(ab-mac+blh)+acy^2(l+d)-abcy(l+d)-bcdx(l-ma)=0\ldots.(6)$.

Its discriminant is $-\frac{1}{4}ab^2c^2dl(l+d)(l-ma)(ab+cd-bdh)$. Rejecting as before the trivial case $h=\frac{ab+cd}{bd}$, this can be zero only if $l=0,-d$, or ma.

Hence (6) will degenerate to a pair of straight lines in just three cases:

If $l=0$, (6) becomes $ad(y-mx)(bx+cy-bc)=0$. Hence if P describes a straight line through O, the locus of R is this same line through O and the line BC, given when P is at O as the indeterminate intersection of BC with itself.

If $l=-d$, (6) becomes $-x\{bdx(ma+d)+dy(mac-ab+bdh)-bcd(ma+d)\}=0$. The second factor gives the equation to the line through C and the second intersection of $y=mx-d$ with the base conic (2). Hence the transform of any line DG through D is CG and the y-axis BD, the latter being given when P is at D as the intersection of BD with an indeterminate line through C.

If $l=ma$, (6) becomes $y\{adx(b-mc+mbh)+acy(d+ma)-abc(d+ma)\}=0$. The second factor gives the equation to the line through B and the second intersection of $y=m(x+a)$ with the base conic (2). Hence the transform of any line AF through A is BF and the x-axis AC, the latter being given when P is at A as the intersection of AC with an indeterminate line through B.

The conic (6) passes through the points O, B', C, and, since every point on the base conic is self corresponding, the points in which $y=mx+l$ intersects the base conic.

The line BC whose equation is $bx+cy-bc=0$ transforms into $b^2dx^2(a+c)+ac^2y^2(b+d)+bcdxy(2a+bh)-b^2cdx(a+c)-abc^2y(b+d)=0$ .....(7). The tangent to it at the origin, $bdx(a+c)+acy(b+d)=0$, passes through the intersection $\left(\frac{ac(b+d)}{ab-dc}, \frac{-bd(a+c)}{ab-dc}\right)$ of AD and BC. Further, (7) is tangent to the base conic (2) at the points B and C. Thus, the tangent at B to either (2) or (7) is $bdx(a-c+bh)+acy(b+d)-abc(b+d)=0$.

Applying (3), the equation to the curve which transforms *into* the line AD, or $dx+ay+ad=0$, is $bd^2x^2(a+c)+a^2cy^2(b+d)+abdxy(2c+dh)+abd^2x(a+c)+a^2cdy(b+d)=0$, which has the same tangent at the origin as (7).

The line at infinity transforms by (5) into the conic $bdx^2+acy^2+bdhxy-bcdx-abcy=0$ ....(8). Since (2) and (8) differ only by the linear expresssion $ad(bx+cy-bc)$, the points of intersection of BC with (2) lie on (8); also (2) and (8) are simultaneously ellipses, parabolas, or hyperbolas. The discriminant of (8), $-\frac{1}{4}ab^2c^2d(ab+cd-bdh)$, shows that it breaks up into two right lines only in trivial case above excluded.

The conic which transforms into the line at infinity, given by the vanishing of the denominator of (3), is $bdx^2+acy^2+bdhxy+abdx+acdy=0$ ....(9) passing through O, A, D. Subtracting (8) from (9), we find their intersections lie on the line $bdx(a+c)+acy(b+d)=0$, which passes through K, the intersection of AD and BC.

Note that the conics (2), (8) and (9) are *similar*

Generally, *the intersections of the conic which transforms into any straight line with the conic into which that straight line transforms lie two on the straight line OK and two on the line itself, the latter two being real and dis-*

tinct, coincident, or imaginary, according as the line intersects the base conic in two real, coincident, or imaginary points.

The conic which transforms into $y=mx+l$, given by substituting from (3) into $y_1=mx_1+l$, is $bdx^2(l+mc)+acy^2(l-b)+bxy(mac-cd+ldh)$ $+abdx(l+mc)+acdy(l-b)=0\ldots.(10)$.

This intersects the conic (6) into which $y=mx+l$ transforms in four points, which, if we subtract (6) from (10) and factor, are seen to lie on

$$\{\, acy(b+d)+bdx(a+c) \,\}\{\, mx-y+l \,\}=0.$$

O is one of the two intersections lying on OK. Call the other H. Then the point in which $y=mx+l$ meets OK and the point H mutually correspond. *We thus have an involution marked out on* OK.

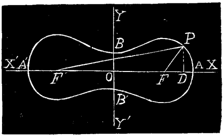
We saw above that the points A, D, O transform into the lines AC, DB, BC respectively. Now we can prove either geometrically or analytically that the lines AD, AO, DO transform into the points O, C, B respectively. Thus the sides and vertices of $\triangle ADO$ transform into the vertices and sides of $\triangle OBC$. With this exception the correspondence between the points in the two systems is one to one. The *projective* treatment of this transformation and its dual will be given elsewhere.

[For Projective Treatment, see my paper in the *Rendiconti del Circolo di Palermo.*]

THE RECTIFICATION OF THE CASSINIAN OVAL BY MEANS OF ELLIPTIC FUNCTIONS.

By F. P. MATZ, Sc. D., Ph. D., Mechanicsburg, Pennsylvania.

The Cassinian Oval is the locus of a point the *product* of whose distances from two fixed points is constant.



Let P be any point on the curve, F and F' the foci, O the middle point of FF', $OD=x$, $DP=y$, $OF=c$, $FP=\rho$, and $F'P=\rho'$; then, according to the definition of the curve, $\rho\rho'=m^2\ldots.(1)$. From the diagram, $\rho=\pm\sqrt{[(x-c)^2+y^2]}$ and $\rho'=\pm\sqrt{[(x+c)^2+y^2]}$; that is, from (1) we obtain the equation,

$$\sqrt{\{\, [(x-c)^2+y^2]\times[(x+c)^2+y^2] \,\}}=m^2\ldots.(a).$$

$\therefore\ (x^2+y^2+c^2)^2-4c^2x^2=m^4\ldots.(2)$; and this is the Cartesian equation of the Cassinian Oval, the co-ordinate axes being rectangular.

Put $OP=r$, and the $\angle POD=(\frac{1}{2}\pi-\theta)$; then $OD=x=r\sin\theta$, and $PD=y=r\cos\theta$. From (2), therefore, we have $r^4+2c^2(1-2\sin^2\theta)r^2=m^4-c^4\ldots.(b)$, or $r^4+(2c^2\cos 2\theta)r^2=m^4-c^4\ldots.(3)$, which is a convenient

form for the *central-polar* equation of the Cassinian Oval.

I. RECTIFICATION.—In order to *rectify* the Cassinian Oval, we deduce, from (3)

$$\cos 2\theta=\frac{(m^4-c^4)-r^4}{2c^2r^2}\ldots.(c).\quad \therefore \sin 2\theta=\sqrt{\left(\frac{4c^4r^4-[(m^4-c^4)-r^4]^2}{4c^4r^4}\right)}\ldots.(d).$$

$$\therefore \frac{d\theta}{dr}=\frac{(m^4-c^4)-r^4}{2c^2r^3\sin 2\theta},\text{ and }\left(\frac{rd\theta}{dr}\right)^2=\frac{[(m^4-c^4)+r^4]^2}{4c^4r^4-[(m^4-c^4)-r^4]^2}\ldots.(e).$$

[The following transformation of (c) affords a *second* method for the derivation of (e):

$$\theta=\tfrac{1}{2}\cos^{-1}\left(\frac{(m^4-c^4)-r^4}{2c^2r^2}\right)=\cos^{-1}\left[\sqrt{\left(\frac{(m^4-c^4)-r^4}{4c^2r^2}+\tfrac{1}{2}\right)}\right]$$

$$=\cos^{-1}\left(\frac{\sqrt{[m^4-(c^2-r^2)^2]}}{2cr}\right),=\sin^{-1}\left(\frac{\sqrt{[(c^2+r^2)^2-m^4]}}{2cr}\right)\ldots.(f).]$$

From (c), when $\theta=0$, we have $r=\pm\sqrt{(m^2-c^2)}$, $=\pm b$; also, when $\theta=\frac{1}{2}\pi$, we have $r=\pm\sqrt{(m^2+c^2)}$, $=\pm a$. Since the perimeter of the Cassinian Oval is composed of four equal quadrantal arcs,

$$\mathsf{P}=4\int_b^a\sqrt{\left(\frac{4c^4r^4-[(m^4-c^4)-r^4]^2+[(m^4-c^4)+r^4]^2}{4c^4r^4-[(m^4-c^4)-r^4]^2}\right)}dr$$

$$=8m^2\int_b^a\frac{r^2dr}{\sqrt{\{[(c^2+r^2)^2-m^4]\times[m^4-(c^2-r^2)^2]\}}}=8m^2.$$

$$\int_b^a\frac{r^2dr}{\sqrt{\{[(m^2+c^2)+r^2]\times[(m^2+c^2)-r^2]\times[r^2+(m^2-c^2)]\times[r^2-(m^2-c^2)]\}}}$$

$$=8m^2\int_b^a\frac{r^2dr}{\sqrt{\{[(m^2+c^2)^2-r^4]\times[r^4-(m^2-c^2)^2]\}}}\ldots.(4).$$

Put $\quad r^4=(m^2+c^2)^2\cos^2\phi+(m^2-c^2)^2\sin^2\phi\ldots.(g);\quad$ then
$4r^3dr=2[-(m^2+c^2)^2+(m^2-c^2)^2]\sin\phi\cos\phi d\phi\ldots.(h).$

$$\therefore r^2dr=\frac{-2m^2c^2\sin\phi\cos\phi d\phi}{[(m^2+c^2)^2\cos^2\phi+(m^2-c^2)^2\sin^2\phi]^{\frac{1}{4}}}\ldots.(i).$$

Transforming (4) by means of (g) and (i), the expression for the required perimeter becomes

$$\mathsf{P}=8m^2\int_o^{\frac{1}{2}\pi}\frac{2m^2c^2[(m^2+c^2)\cos^2\phi+(m^2-c^2)^2\sin^2\phi]^{-\frac{1}{4}}\sin\phi\cos\phi d\phi}{\sqrt{\{[(m^2+c^2)^2-(m^2-c^2)^2]\sin^2\phi\times[(m^2+c^2)^2-(m^2-c^2)^2]\cos^2\phi\}}}$$

$$=4m^2\int_o^{\frac{1}{2}\pi}\frac{d\phi}{[(m^2+c^2)^2\cos^2\phi+(m^2-c^2)^2\sin^2\phi]^{\frac{1}{4}}}$$

$$=4m^2\int_o^{\frac{1}{2}\pi}\frac{d\phi}{\{(m^2+c^2)^2-[(m^2+c^2)^2-(m^2-c^2)^2]\sin^2\phi\}^{\frac{1}{4}}}$$

$$=4m^2\int_o^{\frac{1}{2}\pi}\frac{d\phi}{[(m^2+c^2)^2-4m^2c^2\sin^2\phi]^{\frac{1}{4}}}=\frac{4m^2}{\sqrt{(m^2+c^2)}}\int_o^{\frac{1}{2}\pi}\frac{d\phi}{[1-C^2\sin^2\phi]^{\frac{1}{4}}}$$

$$= \frac{4m^2}{\sqrt{(m^2+c^2)}} \int_0^{\frac{1}{2}\pi} \left[1+\frac{C^2 \sin^2\phi}{4}+\frac{5C^4 \sin^4\phi}{32}+\frac{15C^6 \sin^6\phi}{128}+\text{etc.}\right]d\phi$$

$$=2\pi\sqrt{\left(\frac{m^4}{m^2+c^2}\right)}\left[1+\frac{C^2}{8}+\frac{15C^4}{256}+\frac{75C^6}{2048}+\text{etc.}\right]\ldots.(5),\quad \text{in which}$$

$C^2=4m^2c^2 \,/\, (m^2+c^2)^2$. When $c=2$ and $m^4=25$, we deduce from (5) the numerical result, $P=12.7329+$, expressing the required perimeter. With respect to the *locus* represented by (3), we have the following hypotheses: $m>c\ldots.(\alpha)$, $m=c\ldots.(\beta)$, and $m<c\ldots.(\gamma)$. Under the first hypothesis, the said locus represents the *Cassinian Oval*; under the second hypothesis, the said locus represents the *Bernoullian Lemniscate*; and under the third hypothesis, the said locus represents two *ovaliform figures*.

In rectifying under the hypotheses (β) and (γ), the term $(m^2-c^2)^2$ in (4) and in (g) must be altered accordingly.

[To be continued.]

THE CONSTRUCTION OF THE SUN'S PATH.

By ERIC DOOLITTLE, Professor of Mathematics in the State University of Iowa, Iowa City, Iowa.

In the *Archiv der Mathematik and Physik*, Vol. LIII. Part IV., there is a very interesting article by Professor Hoza on the graphical construction of the sun's apparent path. He considers the earth as stationary in its orbit during a perod of twenty-four hours, and obtains the projection of the apparent path during that time upon the plane of the Equator; the result being, as might be expected, an ellipse, the ratio of whose axes is as 1 : cos δ.

This construction only applies to those places on the earth where the sun actually rises and sets each day, nor is the exact path thus found, since the sun's motion is not taken into account. An investigation of this latter is not difficult; it will lead us to a very interesting spiral curve.

Let us take the vernal equinox as an origin; the arc of the equator as an axis of X positive toward the right, and a great circle perpendicular to the equator through the vernal equinox as the axis of Y. The circles lie on the celestial sphere, whose radius is considered as unity.

Then, if c be the angular velocity of the earth on its axis, and K that of the sun in the ecliptic, (considered as uniform), we will have at a time t after the time of vernal equinox:

$$x = \tan^{-1}\{\cos e \tan Kt\} - ct \ldots (1)$$

$$y = \sin^{-1}\{\sin e \sin Kt\} \ldots (2)$$ where e, the obliquity of the ecliptic, is the constant angle $23^\circ\ 27'$.

Also, consider the point (x, y) as orthogonally projected upon the plane of the equator, and let the radius vector from the center of the sphere to the projected point be ρ. We may write the additional equation,

$$\rho^2 = \cos^2 y = (1 - \sin^2 e \sin^2 Kt) \ldots (3).$$

By eliminating t between (1) and (2) we may obtain the equation of the path in spherial co-ordinates, or with (3) and the formula

$$\rho \sin \varphi = \sin x \ldots (4),$$ we may find the equation of the projection upon the equator in polar co-ordinates. It will be easiest to examine the path by the help of (1), (2), and (3) without eliminating.

By differentiating (1)

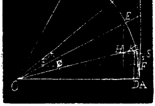
$$\frac{dx}{dt} = \frac{K \cos e}{\sin^2 e \sin^2 Kt + \cos^2 e} - c.$$

The maximum value of the first term of this expression occurs when $t=0$: it is $K \sec e$ or, about $1.1K$. But $c = 366\frac{1}{4}K \ldots (5)$ and hence $\frac{dx}{dt}$ is always minus, and x is always a decreasing function of t.

Similarly, $\frac{dy}{dt} = K \sin e \frac{\cos Kt}{\sqrt{1 - \sin^2 Kt \sin^2 e}}$ and y increases as Kt increases from 0 to $\frac{\pi}{2}$, at which point y attains the maximum value, e. But, by (5), as Kt increases to $\frac{\pi}{2}$, ct increases to $183\frac{1}{8}\pi$, and, from (1), x decreases to $-182\frac{5}{8}\pi$. Thus the spiral begins at the vernal equinox, and reaches its highest point after $91\frac{5}{16}$ revolutions about the sphere.

When Kt increases to π, y decreases to 0, and x decreases to $-365\frac{1}{4}\pi$, so that the spiral turns downward and crosses the equator after $182\frac{5}{8}$ revolutions. The negative values are similar and show that the spiral begins to repeat itself after $365\frac{1}{4}$ revolutions.

Equation (3) shows us that the projection on the plane of the equator is a similar spiral, which begins at the vernal equinox and draws continually nearer the pole for 91 revolutions, after which it widens out until $\rho = 1$ at the autumnal equinox. The minumum value of ρ is $\cos e$, and the curve is always concave toward the pole.



We may make an accurate construction of this projection by points, as follows:

Draw the unit circle and lay off the given angles and their sines:

$AOK = Kt$; $AOE = e$; $AOC = ct$.

Say $OF' = \sin Kt$ and draw $F'F$ and FD.

Then $AOF = y$ (by (2)) and $OD = \cos y = \rho$. Draw HS, meeting the tangent in z. Then AOZ = first term of x and if AZ' be laid back-

ward from C to C', $AOC'=x$. Finally, draw $C'M$ to DF; then $AOM=\varphi$ by (4), and AOM and OD are co-ordinates of points on our spiral. In the figure, AK was taken 10°, and thus $AC=20\pi+62\frac{1}{2}^\circ$.

So accurate a construction might be used sometimes; for instance for the time of sun-rise and sun-set especially near the poles, that is where the spiral cuts the projected Horizon circle. For showing the form of the spiral it can of course be much shortened: $\tan^{-1}(\cos e \tan Kt)$ may be written Kt, and in fact, when it is noticed how very nearly parallel the spires are, many convolutions may be interpolated without computation.

A METHOD OF INTEGRATING CERTAIN DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND N-TH DEGREE.

By RALPH H. KUNSTADTER, Graduate Student, Yale College.

The regular solution of the equation $e^{ax}f(p) \pm e^{by}f'(p)=0$, (where $p=\dfrac{dy}{dx}$) as given in text-books on differential equations is performed after substituting x' far e^{ax} and y' for e^{by}.

In the following demonstration, I shall endeavor to give a very rational and perhaps a natural method which might be easily generalized for similar equations consisting of two members only.

Regarding our equation, we at once recognize it as being a trancendental and exponential equation, which in order to solve we will bring into logarithmic form. For this purpose, multiply $e^{ax}f(p)+e^{by}f'(p)=0 \ldots .(1)$ by $e^{ax}f(p)-e^{by}f'(p)=0$ and we have $e^{2ax}f^2(p)-e^{2by}f'^2(p)=0 \ldots .(2)$.

Applying logarithms, we have $\log e^{2ax}+2\log f(p)=\log e^{2by} +2\log f'(p) \ldots .(3)$. Or $ax+\log f(p)=by+\log f'(p) \ldots .(4)$.

Differentiating this equation, we have

$$adx+\frac{\varphi(p)dp}{f(p)}=bdy+\frac{\varphi'(p)dp}{f'(p)} \ldots .(5).$$

Dividing by dx and clearing of fractions,

$$f(p)f'(p)a+[\varphi(p)f'(p)-f(p)\varphi'(p)]\frac{dp}{dx}+bpf'(p)f(p)=0 \ldots .(6). \quad \text{Or}$$

$$\frac{dp}{dx}[\varphi(p)f'(p)-f(p)\varphi'(p)]+af(p)f'(p)+bpf'(p)f(p)=0.$$

We see that the separation of $\psi(p)dp$ and of dx can be distinctly performed and hence the solution of our problem is theoretically done. Should the given equation be of the form $e^{ax}f(p)-e^{by}f'(p)=0$, we obtain the logarithm of it without multiplying by a factor.

To illustrate our method take the equation,

$$e^{3x}(p-1)+e^{2y}p^3=0; \quad \text{or} \quad e^{3x}\left(\frac{dy}{dx}-1\right)+e^{2y}\left(\frac{dy}{dx}\right)^3=0.$$

We have $e^{6x}(p-1)^2-e^{4y}(p)^6=0$. Applying logarithms and dividing by 2, $3x+2\log(p-1)-2y-3\log p=0$.

Differentiating this equation, we have

$3dx+\dfrac{dp}{p-1}-2dy-3\dfrac{dp}{p}=0$, or $3p^2dx-3pdx+pdp-2p^2dy+2pdy-3pdp$

$+6dp=0$. Dividing through by dx, we get $\dfrac{dp}{dx}(3-2p)+\dfrac{dy}{dx}(2p-2p^2)+3p^2$

$-3p=0$, $\dfrac{dy}{dx}(3-2p)=2p^3+3p-5p^2$, and $\displaystyle\int\frac{dp(3-2p)}{2p^3+3p-5p^2}=x$.

It is not necessary to continue this as it is now to be treated in the customary way.

Similarly, we solve $e^{3x}\left(\dfrac{dy}{dx}-1\right)-e^{2y}\left(\dfrac{dy}{dx}\right)^3=0$.

ARITHMETIC.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

48. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Fifty thousand days preceding Thursday, March 7, 1895, was what date and what day of the week?

I. Solution by P. S. BERG, Apple Creek, Ohio.

Every four years previous to March 7, 1895, excepting the four years of which 1800 was one, contained 1461 days. This number is contained in 50000, 34 times with a remainder of 326 days. Since 1800 was not a leap year the 34 periods or 136 years conducts back to March 6th 1759. 326 days further leads to April 14th 1758.

By referring to a table in Olmsted's Astronomy I find this date to have occurred on Thursday.

II. Solution by S. HART WRIGHT, Ph. D., Penn Yan, New York.

Dividing 50000 by 7 gives 6 remainder, and six days before Thursday falls on Friday, the day of the week required.

Any four consecutive years, containing one bissextile year have 1461 days. $50000 \div 1461$ gives 34 four-year periods, hence there are 34—1 bissextile days, the year 1800 not being a leap-year. $50000-33=49967$ days and $49967 \div 365$, gives 136 years +327 days. (1895+66 days)−(136 years +327 days) gives 1758+104 days= April 14, 1758 the required date, in Gregorian Calender or April 3 in the Julian Calender.

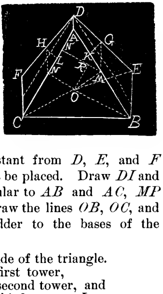
A. L. FOOTE gets as his result Thursday April 13, 1758.

49. Proposed by J. A. CALDERHEAD, B Sc., Superintendent of Schools, Limaville, Ohio.

I have a garden in the form of an equilateral triangle whose sides are 200 feet. At each corner stands a tower; the height of the first tower is 30 feet, the second 40 feet and the third 50 feet. At what distance from the base of each tower must a ladder be placed, that it may just reach the top of each? And what is the length of the ladder, the garden being a horizontal plane?

Solution by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Missouri.

Construction.—Let ABC be the triangular garden and AD, BE, and CF the towers at the corners. Connect the tops of the towers by the lines ED and DF. From G and H, the middle points of DE and DF, draw GM and HN perpendicular to DE and DF, and at M and N draw perdendiculars to AB and AC in the triangle ABC, meeting at O. Then O is equally distant from D and E. For, since M is equally distant from D and E, and MO perpendicular to the plane $ABED$, every point of MO is equally distant from D and E. For a like reason, every point of NO is equally distant from D and F; hence, O their point of intersection, is equally distant from D, E, and F and is, therefore, the point where the ladder must be placed. Draw DI and DJ parallel to AB and AC, GK and HL perpendicular to AB and AC, MP perpendicular to AC and OR parallel to NP. Draw the lines OB, OC, and OA, the required distances from the base of the ladder to the bases of the towers. Draw EO, the length of the ladder.



1. $AB=BC=AC=200$ ft. $=s$, the side of the triangle.
2. $FC=50$ ft. $=a$, the height of the first tower,
3. $EB=40$ ft. $=b$, the height of the second tower, and
4. $AD=30$ ft. $=c$, the height of the third tower. Let
5. $h=\sqrt{[AB^2-(\frac{1}{2}AC)^2]}=\sqrt{[s^2-(\frac{1}{2}s)^2]}=\frac{1}{2}\sqrt{3}s=100\sqrt{3}$ ft. $=$the perpendicular from B to the side AC.
6. $EI=BE-BI(=AD)=(b-c)=40$ ft. -30 ft. $=10$ ft.
7. $GK=\frac{1}{2}(EB+AD)=\frac{1}{2}(b+c)=\frac{1}{2}(40$ ft. $+30$ ft.$)=35$ ft. In the similar triangles DIE and GKM,
8. $DI:IE::GK:KM$, or $s:b-c::\frac{1}{2}(b+c):KM$.

9. $\therefore\ KM=\dfrac{b^2-c^2}{2s}=\dfrac{40^2-30^2}{2\times 200}=1\frac{3}{4}$ ft.,

10. $AM=AK+KM=\frac{1}{2}s+\dfrac{b^2-c^2}{2s}=\dfrac{s^2+b^2-c^2}{2s}=101\frac{3}{4}$ ft., and

11. $BM=AB-AM=s-\dfrac{s^2+b^2-c^2}{2s}=\dfrac{s^2+c^2-b^2}{2s}=98\frac{1}{4}$ ft.

In like manner,

12. $HL=\frac{1}{2}(a+c)=\frac{1}{2}(50\text{ ft.}+30\text{ ft.})=40$ ft.,

13. $LN=\dfrac{a^2-c^2}{2s}=4$ ft.,

14. $AN=AL+LN=\frac{1}{2}s+\dfrac{a^2-c^2}{2s}=\dfrac{s^2+a^2-c^2}{2s}=104$ ft.

15. $NC=AC-AN=s-\dfrac{s^2+a^2-c^2}{2s}=\dfrac{s^2+c^2-a^2}{2s}=96$ ft.

By similar triangles,

16. $AB:AL::AM:AP$, or $s:\frac{1}{2}s::(s^2+b^2-c^2)\div 2s:AP$.

Whence,

17. $AP=(s^2+b^2-c^2)\div 4s=50\frac{7}{8}$ ft.

18. $\therefore\ PL=AL-AP=[\frac{1}{2}s-(s^2+b^2-c^2)\div 4s]=(s^2+c^2-b^2)\div 4s=49\frac{1}{8}$ ft.

A.

19. $RO=PN=PL+LN=(s^2+c^2-b^2)\div 4s+(a^2-c^2)\div 2s=(s^2+2a^2-b^2-c^2)\div 4s=53\frac{1}{8}$ ft. By similar triangles,

20. $AB:BL::AM:MP$, or $s:\frac{1}{2}\sqrt{3}s::(s^2+b^2-c^2)\div 2s:MP$.

Whence,

21. $MP=[(s^2+b^2-c^2)\div 4s]\times\sqrt{3}=50\frac{7}{8}\sqrt{3}$ ft. By similar triangles,

22. $MP:AP::RO:RM$, or $[(s^2+b^2-c^2)\div 4s]\sqrt{3}:(s^2+b^2-c^2)\div 4s::(s^2+2a^2-b^2-c^2)\div 4s:RM$.

23. $RM=(s^2+2a^2-b^2-c^2)4\sqrt{3}s=[(s^2+2a^2-b^2-c^2)\div 12s]\sqrt{3}=17\frac{17}{24}\sqrt{3}$ ft. Again

24. $MP:MA::RO:OM$, or $[(s^2+b^2-c^2)\div 4s]\sqrt{3}:(s^2+b^2-c^2)\div 2s::(s^2+2a^2-b^2-c^2)\div 4s:OM$.

25. $\therefore\ OM=(s^2+2a^2-b^2-c^2)\div 2\sqrt{3}s=[(s^2+2a^2-b^2-c^2)\div 6s]\sqrt{3}=35\frac{5}{12}\sqrt{3}$ ft.

26. $ON=RP=MP-RM=[(s^2+b^2-c^2)\div 4s]\sqrt{3}-(s^2+2a^2-b^2-c^2)\div 12s]\sqrt{3}=[(s^2-a^2+2b^2-c^2)\div 6s]\sqrt{3}=33\frac{1}{6}\sqrt{3}$ ft.

Then

II.

27. $OC=\sqrt{(ON^2+NC^2)}=\sqrt{\left[\left(\dfrac{s^2-a^2+2b^2-c^2}{6s}\sqrt{3}\right)^2+\left(\dfrac{s^2+c^2-a^2}{2s}\right)^2\right]}=\sqrt{[(33\frac{1}{6}\sqrt{3})^2+96^2]}=\sqrt{12516\frac{1}{12}}=111.8796+$ ft.

28. $OA=\sqrt{(ON^2+AN^2)}=\sqrt{\left[\left(\dfrac{s^2-a^2+2b^2-c^2}{6s}\sqrt{3}\right)^2+\left(\dfrac{s^2+c^2-a^2}{2s}\right)^2\right]}=\sqrt{[(33\frac{1}{6}\sqrt{3})^2+104^2]}=\sqrt{14116\frac{1}{12}}=118.8111+$ ft.

29. $OB=\sqrt{(OM^2+MB^2)}=\sqrt{\left[\left(\dfrac{s^2+2a^2-b^2-c^2}{6s}\sqrt{3}\right)^2+\left(\dfrac{s^2+c^2-b^2}{2s}\right)^2\right]}=\sqrt{[(35\frac{5}{12}\sqrt{3})^2+(98\frac{1}{4})^2]}$

1. B. $=\frac{1}{4}\sqrt{214657\frac{1}{3}}=115.8278+$ ft.
$OE=\sqrt{(BE^2+OB^2)}=\sqrt{[(\frac{1}{4}\sqrt{214657\frac{1}{3}})^2+40^2]}$,
$=\sqrt{(13416\frac{1}{2}+1600)}=\sqrt{15016\frac{1}{12}}=122.5402+$ ft. = the length of the ladder.

III. ∴
1. 111.8796 + ft. = the distance from base of the ladder to the base of the tower FC,
2. 118.8111 + ft. = the distance from the base of the ladder to the base of the tower AD.
3. 115.8278 + ft. = the distance from the base of the ladder to the base of the tower BE, and
4. 122.5402 + ft. = the length of the ladder.

[From *Finkel's Mathematical Solution Book*, *p.* 299.]

[NOTE.—This method of solution may be easily extended to the more general case, viz., when the triangle is scalene.]

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

44. Proposed by LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago.

Find the general term in the series 1, 3, 10, 35, 126, 462, 1716, 6435, 24310, ..., which plays a remarkable part in some recent theorems in my Theory of Regular Polygons.

Solution by the PROPOSER.

This series is a "diagonal" in the Triangle of Pascal, as shown in the following table:—

C	0	1	2	3	4	5	6	7	8	
0	1									
1	1	**1**								
2	1	2	1							
3	1	3	**3**	1						
4	1	4	6	4	1					
5	1	5	10	**10**	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	**35**	21	7	1		
8	1	8	28	56	70	56	28	8	1	
9	1	9	36	84	126	**126**	84	36	9	1

Since the mth term in the series lies at the intersection of column m with

row $(2m-1)$, it is $\dfrac{(2m-1)(2m-2)\ldots.(2m-m)}{1.2.3\ldots.m}$

$$=\frac{(2m-1)(2m-2)\ldots.(m+1)}{1.2.3\ldots.(m-1)}=\frac{(2m)!}{2(m\,!)^2}.$$

The part played by the series in the Theory of Regular Polygons is indicated in an article in the current issue of the *Annals of Mathematics.*

Also solved by *A. H. BELL,* and *Professor J. F. W. SCHEFFER.*

45. **Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.**

Find x from $\cos^{-1}\dfrac{1-x^2}{1+x^2}+\tan^{-1}\dfrac{2x}{1-x^2}=\dfrac{4\pi}{3}$.

I. Solution by T. W. PALMER, A. M., Professor of Mathematics, University of Alabama; H. W. DRAUGHON, Olio, Mississippi; A. L. FOOTE, C. E., Middlebury, Connecticut; and the PROPOSER.

Let $\tan\theta=x$. Then $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}, =\dfrac{2x}{1-x^2}$; and $\dfrac{1-x^2}{1+x^2}$

$=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}, =\cos^2\theta(1-\tan^2\theta), =(\cos^2\theta-\sin^2\theta), =\cos 2\theta.$

$$\therefore \cos^{-1}\frac{1-x^2}{1+x^2}+\tan^{-1}\frac{2x}{1-x^2}, =\cos^{-1}\cos 2\theta+\tan^{-1}\tan 2\theta=\frac{4\pi}{3}.$$

$$\therefore 2\theta+2\theta, =4\theta, =\frac{4\pi}{3}, \quad \therefore \theta=\frac{\pi}{3}, =60^\circ.$$

$$\therefore x=\tan\theta, =\tan\frac{\pi}{3}, =\tan 60^\circ, =\sqrt{3}.$$

$\therefore x=\sqrt{3}.$

II. Solution by JOHN B. FAUGHT, A. B., Indiana University, Bloomington, Indiana; J. A. JOHNSON, Jr., Student of the Sophomore Class, University of Mississippi; P. S. BERG, Apple Creek, Ohio; and J. W. WATSON, Middle Creek, Ohio.

$$\text{Since } \tan^{-1}\frac{2x}{1-x^2}=\cos^{-1}\frac{1}{\sqrt{1+\dfrac{4x^2}{(1-x^2)^2}}}=\cos^{-1}\frac{1-x^2}{1+x^2};$$

$$\therefore 2\cos^{-1}\frac{1-x^2}{1+x^2}=\frac{4}{3}\pi, \text{ or } \cos^{-1}\frac{1-x^2}{1+x^2}=\frac{2}{3}\pi=\cos^{-1}(-\tfrac{1}{2}).$$

$\therefore \dfrac{1-x^2}{1+x^2}=-\tfrac{1}{2}$, whence $x^2=3$, and $x=\pm\sqrt{3}$.

Alfo solved by *F. P. MATZ, J. SCHEFFER, C. D. SCHMITT, E. L. SHERWOOD, M. C. STEVENS, G. B. M. ZERR;* and ————

PROBLEMS.

54. Proposed by E. W. MORRELL, Department of Mathematics, Montpelier Seminary, Montpelier, Vermont.

Transform $x^4+y^4+z^4-2y^2z^2-2z^2x^2-2x^2y^2-$ into a product.

[*Bowser's Trigonometry.*]

55. Proposed by MARCUS BAKER, M. A., U. S. Geological Survey, Washington, D. C.

Two right triangles ABC and ABD are so placed as to have one side $x(=AB)$ in common. From P the intersection of their hypotenuses is drawn c perpendicular to x. Knowing the hypotenuses $a=39$ feet and $b=25$ feet and the perpendicular $c=12\frac{6}{7}$ feet, find x. Note this theorem $\frac{1}{m}+\frac{1}{n}=\frac{1}{c}$ or $\frac{1}{\sqrt{a^2-x^2}}+\frac{1}{\sqrt{b^2-x^2}}=\frac{1}{c}$, where m and n are the altitudes of the two triangles, respectively. Also find locus of P. Discuss the case when the triangles are general (not right angled.)

[The same problem, in the form of "two poles" with ropes stretched from top of one to foot of other and the same data given, was contributed by H. C. Wilkes. ED.]

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

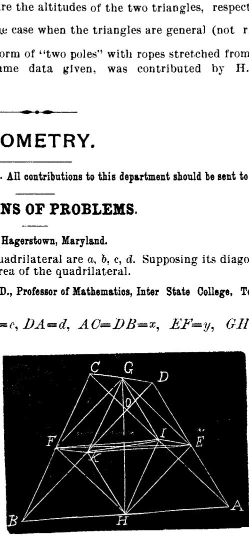
SOLUTIONS OF PROBLEMS.

43. Proposed by J. F. W. SCHEFFER, Hagerstown, Maryland.

The consecutive sides of a quadrilateral are a, b, c, d. Supposing its diagonals to be equal, find them and also the area of the quadrilateral.

Solution by G. B. M ZERR, A. M., Ph. D., Professor of Mathematics, Inter State College, Texarkana, Texas.

Let $AB=a$, $BC=b$, $CD=c$, $DA=d$, $AC=DB=x$, $EF=y$, $GH=z$; also, let H, F, G, E, I, K be the middle points of AB, BC, CD, DA, AC, DB, respectively.



Then $EGFH$, $GKHI$, $FKEI$ are all parallelograms; but $IG=HK=\frac{1}{2}d$, $HI=KG=\frac{1}{2}b$, $EI=KF=\frac{1}{2}c$, $FI=EK=\frac{1}{2}a$, $EG=GF=FH=HE=\frac{1}{2}x$.

$\therefore \frac{1}{2}(a^2+c^2)=y^2+$

$IK^2 \ldots.(1)$, $\frac{1}{2}(b^2+d^2)=z^2+IK^2 \ldots.(2)$, $x^2=y^2+z^2 \ldots.(3)$.

$(1)-(2)$ gives, $\frac{1}{2}(a^2+c^2-b^2-d^2)=y^2-z^2 \ldots.(4)$.

From (3) and (4), $y=z$, and $a^2+c^2=b^2+d^2 \ldots.(5)$,

$(a^2-b^2)^2=(c^2-d^2)^2 \ldots.(6)$.

$\therefore$ $EGFH$ is a square and AC is perpendicular to BD.

$\therefore$ area $ABCD=\frac{1}{2}x^2$.

$\frac{1}{2}x^2=\frac{1}{2}bx \sin ACB+\frac{1}{2}cx \sin ACD$

$$\sin ACB=\frac{\sqrt{4b^2x^2-(x^2+b^2-a^2)^2}}{2bx},\quad \sin ACD=\frac{\sqrt{4c^2x^2-(x^2+c^2-d^2)^2}}{2cx}.$$

$$\therefore\ 2x^2=\sqrt{4b^2x^2-(x^2+b^2-a^2)^2}+\sqrt{4c^2x^2-(x^2+c^2-d^2)^2} \ldots.(7).$$

From (5), (6), and (7) $\quad 2x^4-2x^2(a^2+c^2)+(a^2-d^2)^2+(c^2-d^2)^2=0$.

$$\therefore\ x^2=\tfrac{1}{2}\left\{a^2+c^2\pm\sqrt{(a^2+c^2)^2-2(a^2-d^2)^2-2(c^2-d^2)^2}\right\}.$$

If $a=b=c=d$, $x^2=2a^2$.

This problem was also solved by *F. P. MATZ*, *J. C. CORBIN*, and *J. F. W. SCHEFFER*.

44. Proposed by I J. SCHWATT, Ph D., Professor of Mathematics. University of Pennsylvania, Philadelphia, Pennsylvania.

(1). If from the middle point M of the side BC of the triangle ABC a parallel to the external bisector AF of the angel BAC is drawn to meet AB at K, the point K divides then the side AB in KA

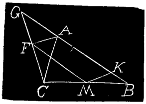
$$=\tfrac{1}{2}(AB+AC) \text{ and } KB=\tfrac{1}{2}(AB-AC).$$

(2). If K is joined to the extremity D of the diameter perpendicular to BC then is KD perpendicular to AB.

Solution by the PROPOSER.

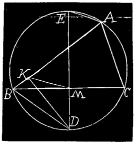
(1.) If from the middle point M of the side BC of the triangle ABC a parallel to the external bisector AF of the angle BAC is drawn to meet AB at K then is the side AB divided in $KA=\frac{1}{2}(AB+AC)$ and KB

$=\frac{1}{2}(AB-AC)$.



From M draw a parallel to AB intersecting the external bisector of BAC at F. Join C to F and produce CF to meet AB at G. Since $CM=MB$ and MF is parallel to AB, therefore is $CF=FG$; The triangles CAF and FAG are equal and therefore is $AG=AC$. $MF=AK=\frac{1}{2}(AB+AG)=\frac{1}{2}(AB+BC)$. $\quad BK=AB-\frac{1}{2}(AB+BC)=\frac{1}{2}(AB-BC.)$

(2.) If K is joined to the extremity D of the diameter perpendicular to BC then is KD perpendicular to AB.



The internal and external bisectors of an angle are perpendicular to each other. Since AD is the internal bisector of the angle BAC therefore is EA the external bisector of the same angle. By (1) is MK parallel to AE or angle EAB=angle AKM; but angle EAB equals angle EDB therefore is angle

AKM equal to angle EDB or the points B, D, M and K are concyclic and therefore is angle BMD=angle $BKD=90^\circ$.

This problem was also solved by *G. B. M ZERR*, *O. W. ANTHONY*, *E. W. MORRELL*, and *P. S. BERG*.

PROBLEMS.

48. Proposed by I. J. SCHWATT, Ph. D., Professor of Mathematics in the University of Pennsylvania, Philadelphia, Pennsylvania.

The Simson line belonging to one point of intersection of Brocord's Diameter, of a triangle with the circumcircle of this triangle is either parallel or perpendicular to the bisector of the angle formed by the side BC of the triangle ABC and the corresponding side B' C' of Brocard's triangle.

49. Proposed by J. C. WILLIAMS, Rome, New York.

Of all triangles inscribed in a given segment of a circle, with the chord as base, the isosceles is the maximum.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

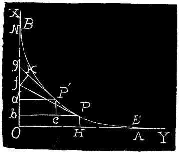
SOLUTIONS OF PROBLEMS.

34. Proposed by GEORGE LILLEY, Ph. D., LL. D., Park School, Portland, Oregon.

A hare is at O, and a hound at E, 40 rods east of O. They start at the same instant each running with uniform velocity. The hare runs north. The hound runs directly toward the hare and overtakes it at N, 320 rods from O. How far did the hound run?

I. Solution by M. C. STEVENS, A. M., Professor of Mathematics, Purdue University, Lafayette, Indiana; COOPER D. SCHMITT, A. M., University of Tennessee, Knoxville, Tennessee; and G. B. M. ZERR, A. M., Ph. D., Inter States College, Texarkana, Texas.

Let u= velocity of hound, v= velocity of hare, $AO=a$, $BO=b$, $OH=x$, $PH=y$, $AP=s$, $\frac{v}{u}=n$. Then $nPA=Of$. $\therefore$ $ns=y-x\frac{dy}{dx}$.



Differentiating with respect to x, we have $n\frac{ds}{dx}=-x\frac{d^2y}{dx^2}$. But s increases as x diminishes, whence $\frac{ds}{dx}=-\sqrt{1+\left(\frac{dy}{dx}\right)^2}$. $\therefore$ $\frac{n}{x}$

$$=\frac{\frac{d^2y}{dx^2}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}.$$ Integrating and noting that $x=a$, $\frac{dy}{dx}=0$, together, n

$\log\frac{x}{a}=\log\left[\sqrt{1+\left(\frac{dy}{dx}\right)^2}+\frac{dy}{dx}\right]$. $\therefore \left(\frac{x}{a}\right)^n=\sqrt{1+\left(\frac{dy}{dx}\right)^2}+\frac{dy}{dx}$, also by reciprocals, $\left(\frac{a}{x}\right)^n=\sqrt{1+\left(\frac{dy}{dx}\right)^2}-\frac{dy}{dx}$. Addding we get $2\sqrt{1+\left(\frac{dy}{dx}\right)^2}=2\frac{ds}{dx}$

$$=\left(\frac{x}{a}\right)^n+\left(\frac{a}{x}\right)^n. \quad \therefore s=\tfrac{1}{2}\int_0^a\left[\left(\frac{x}{a}\right)^n+\left(\frac{a}{x}\right)^n\right]dx=\tfrac{1}{2}\left[\frac{x^{n+1}}{a^n(n+1)}+\frac{a^nx^{1-n}}{1-n}\right]_0^a$$

$$=\frac{a}{1-n^2}. \quad \therefore s=\frac{a}{1-n^2}, \text{ but } ns=b, \quad \therefore n=\frac{b}{s}. \quad \therefore s=\frac{as^2}{s^2-b^2}, \quad \therefore s^2-as=b,$$

$s=\frac{1}{2}[a\pm\sqrt{(a^2+4b^2)}]$ $s=\frac{1}{2}[a+\sqrt{(a^2+4b^2)}]$, the minus sign not being admissable. $s=\frac{1}{2}[40+\sqrt{(1600+409600)}]=340.624+$rods. To find the rectangular equation to the curve, we have by subtracting $2\frac{dy}{dx}=\left(\frac{x}{a}\right)^n-\left(\frac{a}{x}\right)^n$. $\therefore 2(y+C)$

$=\frac{x^{n+1}}{a^n(n+1)}-\frac{a^nx^{1-n}}{1-n}$. When $y=0$, $x=a$, which gives $C=-\frac{an}{1-n^2}$.

$$\therefore 2\left(y-\frac{an}{1-n^2}\right)=\frac{x^{n+1}}{a^n(n+1)}-\frac{a^nx^{1-n}}{1-n}.$$

II. **Solution by ALFRED HUME, C. E., D. Sc., University of Mississippi, and the PROPOSER.**

The tangent to the path of the hound always passes through the position of the hare, the point of tangency being the simultaneous position of the hound. Let P and P' be two positions of the hound, PP' being infinitesimal, f and g corresponding positions of the hare. Draw the ordinates $P'd$ and Pb and the perpendiculars fk and $P'e$ to $P'g$ and Pb respectively. Let $AP=s$, $PP'=ds$, $OB=x$, $bd=eP'=dx$, $OF=z$, $fg=dz$. Now, since the two animals run with uniform velocities, $\frac{ds}{dz}=$ some constant$=l$. In the limit, $\triangle$'s $P'eP$ and fgk are similar.

$$\therefore \frac{ds}{dx}=\frac{dz}{gk};\quad gk=dx\frac{dz}{ds}=\frac{dx}{l}.$$

The change in the length of the tangent as the hound runs from P to P' is due to a positive increment, gk, at one end, and a negative increment, PP', at the other. Therefore, if $t=$ the length of the tangent, $dt=\frac{dx}{l}-ds$, and, integrating, $t=\frac{x}{l}-s+C$. When $t=40$, $x=0$, and $s=0$; $\therefore C=40$. For

s substitute its value lz; then $t=\frac{x}{l}-lz+40$. When $t=0$, $x=320$, and $z=320$.

$$\therefore 0=\frac{320}{l}-320l+40, \text{ and } l=1.0644+. \quad s=lz=1.0644\times 320=340.$$

624+, the distance in rods that the hound runs.

Also solved by *A. L. FOOTE*, *P. S. BERG* and *F. P. MATZ*.

35. **Proposed by H. C. WHITAKER, B. S., C. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.**

At the rate of 10 cubic inches per second, water is running into a vessel in the shape of a right conic frustum, the radii of whose upper and lower bases are respectively 15 and 10 inches, and whose altitude is 20 inches. At what rate per second is the *depth* of the water increasing, when *it* is exactly 8 inches?

I. **Solution by J. C. NAGLE, M. A., M. C. E., Professor of Civil Engineering, A. and M. College, College Station, Texas, and ALFRED HUME, C. E., D. Sc., University of Mississippi.**

Represent the depth and the volume of water at any instant by h and V respectively. The radius of the free surface of the water will be $10+\frac{h}{4}$.

Therefore $V=\frac{\pi}{3}h\left[100+\left(10+\frac{h}{4}\right)^2+10\left(10+\frac{h}{4}\right)\right]$.

Differentiating, $dV=\frac{\pi}{3}(300+15h+\frac{3}{16}h^2)$. Substituting for dV and h 10 and 8 respectively, $dh=\frac{5}{72\pi}$, or the depth is increasing at the rate of $\frac{5}{72\pi}=0.02210484+$ of an inch per second.

[This result may also be obtained as follows: The area of the free surface of the water at the instant under consideration is 144π. Therefore the depth is increasing at the rate of $\frac{10}{144\pi}$ inches per second.]

II. **Solution by Professor J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.**

Let the radii of upper and lower bases be a and b, the altitude$=h$, $x=$ the distance of water from the lower base after the time t, y the radius of section at the distance x; then, denoting by c the quantity of water flowing in per second, $\pi y^2dx=cdt$.

$$\therefore \frac{dx}{dt}=\frac{c}{\pi y^2}=\frac{ch^2}{\pi[(a-b)x+bh]^2}.$$

For the given numerical values, we have $\frac{dx}{dt}=\frac{5}{72\pi}$=required rate at distance $=8$.

Also solved by *P. S. BERG*, *F. P. MATZ* and *E. L. SHERWOOD*.

PROBLEMS.

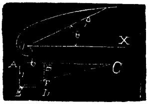
43. **Proposed by Professor C. E. WHITE, A. M., Trafalgar, Indiana.**

Prove that $\int_2^1 \frac{x^{a-1}+x^{-a}}{1+x} \cdot \frac{dx}{x} = \log\left(\tan\frac{a\pi}{2}\right)$, when $a>0$ and <1.

[*Williamson's Int. Calculus, p. 154.*]

44. **Proposed by DE VOLSON WOOD, C. E., Professor of Mechanical and Electrical Engineering in Stevens Institute of Technology, Hoboken, New Jersey.**

Find the equation of a curve in which $\rho=f(\theta)$, in which ρ is equal to BC, an intercept of any secant drawn from the corner E of the rectangle $AEDB$, and prolonged to cut AB prolonged in C. Let equal increments of θ be proportional to the equal increments of DB as divided by the secant EF, θ being zero when EC coincides with ED, and $\theta=2\pi$ when EF passes through B. Determine the asymptotes.



[Prof. Mac Cord of Stevens Institute, while investigating the curve of intersection of a plane with the surface of a certain volume, found that it had the property of the above problem, and he referred it to Professor Wood to investigate in regard to asymptotes. If the curve is not known to science, Professor Mac Cord desires to christen it as "The Thistle of Scotland."]

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

23. **Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi University Post Office, Mississippi.**

A heavy particle is placed upon the surface $\sqrt{x}+\sqrt{y}+\sqrt{z}=c$. The axis of z being vertical and the coefficient of friction being $\sqrt{2}$, show that a point of equilibrium (all friction possible being brought into action) z is a harmonical mean between x and y.

I. **Solution by WILLIAM HOOVER, A. M., Ph. D., Member of London Mathematical Society and Professor of Mathematics, Ohio University, Athens, Ohio.**

If X, Y, Z be the forces of which P is the resultant independent of the reaction of the surface on the particle parallel to the co-ordinate axes, $\mu=\sqrt{2}=$ the coefficient of friction, and $\mu=0$ the equation to the surface, for limiting equilibrium we have

$$\left(X\frac{du}{dx}+Y\frac{du}{dy}+Z\frac{du}{dz}\right)^2 \div P^2\left(\frac{du^2}{dx^2}+\frac{du^2}{dy^2}+\frac{du^2}{dz^2}\right)=\frac{1}{1+\mu^2}\;\ldots.(1),$$

the derivatives being partial.

We have $u=\sqrt{x}+\sqrt{y}+\sqrt{z}-c=0\;\ldots.(2)$,

whence $\dfrac{du}{dx}=\dfrac{1}{2\sqrt{x}},\ \dfrac{du}{dy}=\dfrac{1}{2\sqrt{y}},\ \dfrac{du}{dz}=\dfrac{1}{2\sqrt{z}}\ldots(3).$

Also, $X=Y=0,\quad Z=-g=P\ldots(4).$

Substituting in (1) and reducing, $z=\dfrac{2xy}{x+y}\ldots(5).$

II. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi.

If W is the weight of the particle, N and T its normal and tangential components,

$$W^2=N^2+T^2.$$

Also, when the particle is on the point of sliding,

$$T=\sqrt{2}\,N.$$

Hence $W^2=3N^2$.

Again $W\cos\theta=N$, θ being the angle between the normal and the Z-axis.

Now $\cos\theta=\dfrac{\dfrac{dF}{dz}}{\sqrt{\left(\dfrac{dF}{dx}\right)^2+\left(\dfrac{dF}{dy}\right)^2+\left(\dfrac{dF}{dz}\right)^2}}$, $F(x,y,z)=0$ being the equation of the surface, and the differential-coefficients being partial.

$$\frac{dF}{dx}=\frac{1}{2\sqrt{x}},\ \frac{dF}{dy}=\frac{1}{2\sqrt{y}},\ \frac{dF}{dz}=\frac{1}{2\sqrt{z}};$$

and, therefore, $\cos\theta=\dfrac{\dfrac{1}{\sqrt{z}}}{\sqrt{\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}}}$.

But $W^2\cos^2\theta=N^2=\dfrac{W^2}{3}$, from which $3\cos^2\theta=1$.

Substituting, $\dfrac{3}{z}=\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}$, or $\dfrac{2}{z}=\dfrac{1}{x}+\dfrac{1}{y}$, and z is a harmonical mean between x and y.

Solutions of this problem were also received from *F. P. MATZ* and *G. B. M. ZERR*.

24. Proposed by J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

A sphere whose center of gravity does not coincide with its geometrical center is placed on a rough inclined plane. State under what circumstances the sphere will slide without rolling, roll without sliding, and neither roll nor slide.

Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics, Irving College, Mechanicsburg, Pennsylvania.

On a horizontal plane, the sphere will neither roll nor slide; but it will *rock* about the vertical drawn through the point of support. Down an inclined plane, the sphere will roll without sliding, until the *initial horizontal plane* through which the centroid has (by the rolling) become a *vertical* plane. So

long as this initial horizontal plane remains a vertical plane, the sphere will slide without rolling.

25. Proposed by Professor GEORGE LILLEY, LL. D., Ex-President of Washington State Agricultural College and School of Science, Portland, Oregon.

It is known that if the velocity of a certain freight train is 30 miles an hour it can be brought to a stand still in a distance of 500 feet by setting the brakes. It was stopped in 1200 feet by setting the brakes. Find its velocity, the forces exerted by the brakes being the same in each case.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Physics, Inter State College, Texarkana, Texas, and the PROPOSER.

$\frac{1}{2}Mv^2=Rs$, where M=mass, v=velocity, R=resistance of brakes, s=distance train runs after setting brakes.

30 miles per hour=44 feet per second.

$\therefore \frac{1}{2}M(44)^2=500R\ldots.(1).\quad \frac{1}{2}Mv^2=1200R\ldots.(2).$

$$(2)\div(1),\quad \left(\frac{v}{44}\right)^2=\frac{12}{5}.\quad \therefore 5v^2=12(44)^2.$$

$\therefore v=88\sqrt{\frac{3}{5}}$ feet per second $=60\sqrt{\frac{3}{5}}$ =46.4758 miles per hour.

Also solved by *F. P. MATZ* and *E. W. MORRELL.*

PROBLEMS.

31. Proposed by O. W. ANTHONY, Mexico, Mo.

A perfectly elastic but perfectly rough sphere of mass M and radius R, rotating in a vertical plane with an angular velocity of ω, is let fall from a height, a, upon a perfectly elastic but perfectly rough horizontal plane. Determine the motion of the body after striking the plane. What will be its ultimate motion?

32. Proposed by OTTO CLAYTON, A. B., Fowler, Indiana.

The wheel of a wind pump has 60 fans, each turned at an angle 45° to the direction of the axis, and each having 150 square inches exposed to the wind. If the wind blows with velocity V and the wheel rotates with velocity ω what is the component of force or pressure along the axis if it is turned at an angle α to the direction of the wind assuming the resistance the wheel meets in turning to be R?

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

25. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find, if possible, integral values of each of the seven linear measurments of a

rectangular parallolopiped; i. e. length, breadth, height, the diagonals of each of the three different rectangular sides, and the diagonal from an upper corner to the opposite lower corner; or, find integral values, if possible, of a, b, c, d, e, f, and g, as shown in the equations,— $a^2+b^2=c^2$, $a^2+d^2=e^2$, $a^2+f^2=g^2$, $b^2+d^2=f^2$, $b^2+e^2=g^2$, $c^2+d^2=g^2$, $c^2+e^2=f^2$. If not possible, how many of them can have integral values? and which?

Solution by G. B. M ZERR, A. M., Ph. D., Vice-President and Professor of Mathematics and Sciences, Inter State College, Texarkana, Texas.

Let the length, breadth, and height be, $a=8mn(m^4-n^4)$, $b=2mn\{10m^2n^2-3(m^4+n^4)\}$, $c=(m^2-n^2)(m^4+n^4-14m^2n^2)$.

The method of obtaining the above values has been published in several journals and need not be repeated here. From the above we easily get $a^2+b^2=\{2mn(5m^4-6m^2n^2+5n^4)\}^2$, $a^2+c^2=(m^6+17m^4n^2-17m^2n^4-n^6)^2$, $b^2+c^2=(m^6+3m^4n^2+3m^2n^4+n^6)^2=(m^2+n^2)^6$, and $a^2+b^2+c^2=64m^2n^2(m^4-n^4)^2+(m^2+n^2)^6$. This last is a square when $64m^2n^2(m^2-n^2)^2+(m^2+n^2)^4=\square$. Let $m=pn$. Then must $64p^2(p^2-1)^2+(p^2+1)^4=p^8+68p^6-122p^4+68p^2+1=\square$.

I have not yet succeeded in making this last a square. The edges and diagonals of sides, are integral, satisfying six of the relations.

Let $m=2$, $n=1$; then $a=240$, $b=44$, $c=117$, $\sqrt{a^2+b^2}=244$, $\sqrt{a^2+c^2}=267$, $\sqrt{b^2+c^2}=125$, $\sqrt{a^2+b^2+c^2}=5\sqrt{2929}$.

26. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathemetics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find (1) a *square fraction* the arithmetical difference of whose terms is a *cube*; and (2) find a *cubic fraction* the arithmetical sum of whose terms is a *square*.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

(1) Let $\dfrac{y^2}{x^2}$ equal the *square fraction*. Then $x^2-y^2=a$ *cube*$=a^3$. Then $(x+y)(x-y)=a^3$. Put $x+y=a^2$, and $x-y=a$.

$$\text{Then } x=\frac{a^2+a}{2}=\frac{a(a+1)}{2}; \text{ and } y=\frac{a^2-a}{2}=\frac{a(a-1)}{2}.$$

Whatever integral values be assigned to a, x and y will always be integral; for whether a is even or odd, $a(a+1)$ and $a(a-1)$ are *even*.

$$\text{Since } x-y=a,\quad \frac{a(a-1)}{2}+a=\frac{a(a+1)}{2}.$$

$\therefore$ the denominator of the fraction $\dfrac{y}{x}$ is always a more than the numerator. Also, as $\dfrac{a(a-1)}{2}$ is the sum of the series $(1+2+3+\dots+a-1)$, $\dfrac{y}{x}=(1+2+3+\dots+a-1)\div[(1+2+3+\dots+a-1)+a]$, or $(1+2+3+\dots+a-1)\div(1+2+3+\dots+a)$. Putting a equal, consecutively, to the successive

integers beginning with *unity*, we have, respectively, $\frac{y}{x}=\frac{0}{1}, \frac{1}{3}, \frac{3}{6}, \frac{6}{10}, \frac{15}{21},$

$\frac{21}{28}, \frac{28}{36}, \frac{36}{45},$ etc., *ad infinitum*.

(2) Let $\frac{m^3}{n^3}$ be the *cubic fraction*. Then $n^3+m^3=\square=b^2$. Then $(n+m)(n^2-nm+m^2)=b^2$. Put $n+m=n^2-nm+m^2=b$.

$$\text{Then } n=\tfrac{1}{2}\left(b+\sqrt{\frac{b(4-b)}{3}}\right), \text{ and } m=\tfrac{1}{2}\left(b-\sqrt{\frac{b(4-b)}{3}}\right).$$

The only integral values of b that will render $\sqrt{\frac{b(4-b)}{3}}$ rational, are 3 and 4. Whence the respective values of n are 2 and 2, and those of m are 1 and 2. $\therefore \frac{m}{n}=\frac{1}{2}$ or $\frac{2}{2}$.

By putting $n+m=\frac{b}{2}$, and $n^2-nm+m^2=2b$ we obtain $\frac{m}{n}=\frac{0}{4}$ and $\frac{4}{8}$.

Other relations of the factors, both in (1) and (2), will yield other results.

II. Solution by **E. L. SHERWOOD, A. M., Professor of Mathematics and Science, Mississippi Normal College, Houston, Mississippi.**

1st. Let $(x+1)^2=$ one of the terms, and $(a-x)^2$ the other. Then $(a-x)^2-(x+1)^2=$a cube$=(x+1)^3$. Finding a in terms of x, we get, $a=x+(x+1)\sqrt{(x+2)}$, and substituting in the first equation, $[(x+1)\sqrt{(x-2)}]^2-(x+1)^2=(x+1)^3$. Now substitute n^2-2 for x and the last equation becomes $(n^2-1)^2n^2-(n^2-1)^2=(n^2-1)^3$.

$\therefore$ Fraction$=\frac{n^2(n^2-1)^2}{(n^2-1)^2}$, or $\frac{(n^2-1)^2}{n^2(n^2-1)^2}$; difference$=(n^2-1)^3$, in which n may be any integer.

2nd. Since both terms must be cubes, we must have

$$n^3(n^3+1)^3+(n^3+1)^3=\text{a square}=(n^3+1)(n^3+1)^3.$$

$\therefore$ Fraction$=\frac{n^3(n^3+1)^3}{(n^3+1)^4}$, or $\frac{(n^3+1)^3}{n^3(n^3+1)^3}$; Sum$=(n^3+1)^4$, in which n may be any integer

III. Solution by **H. C. WILKES, Murrayville, West Virginia, and A. H. BELL, Hillsboro, Illinois.**

Since when the first number is unity, the sum of any number of successive cubes is a square, if we let $n=$ a root of any cube, then

$$\left[n\left(\frac{n+1}{2}\right)\right]^2-\left[n\left(\frac{n-1}{2}\right)\right]^2=n^3.$$

Let $n=2$, then $3^2-1^2=2^3$,

" $n=3$, then $6^2-3^2=3^3$,

" $n=4$, then $10^2-6^2=4^3$, or in the series 1, 3, 6, 10, 15, etc., we have the difference of the squares of any two contiguous terms equal a cube.

Second case. Let $\frac{y^3}{x^3}$ be the fraction. Then $x^3+y^3=n^2$, or $(x+y)(x^2-xy+y^2)=n^2$. If $(x+y)$ be a square, then x^2-xy+y^2 will be a square. This is only possible when $x=y$. $\therefore$ the sum of any two equal cubes, the sum of whose roots is a square, will be a square, as $\frac{8^3}{8^3}$, or $\frac{512}{512}$, will be an improper cubic fraction the sum of whose terms will be a square. If $x+y=x^2-xy+y^2$, then x^3+y^3 will be a square. This is only possible when $x=2$, $y=1$, and the proper fraction $\frac{1}{8}$ will be cubic and the sum of the terms a square number.

$\therefore (2^3+1^3)=3^2$; also $2^6(2^3+1^3)=24^2$, etc.

Also solved by *P. S. BERG*, *A. L. FOOTE*, *G. B. M. ZERR*, and the *PROPOSER*.

PROBLEMS.

34. Proposed by R. H. YOUNG, West Sunbury, Pennsylvania.

Prove (1) that $\frac{n(n+1)(2n+1)}{6}$ is a whole number for all values of n; and (2) prove that $\frac{(n-1)n(n+1)}{24}$ is a whole number when n is odd.

35. Proposed by G. B. M. ZERR, A. M., Ph. D, Vice-President and Professor of Mathematics and Sciences, Inter State College, Texarkana, Texas.

Decompose into the sum of two squares the number $13^2.61^3$.

36. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find the first six integral values of n in $\frac{n(n+1)}{2}=\square$.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

19. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Find the average area of the circle which is the locus of the middle points of all chords passing through a point taken at random in the surface of a given circle.

II. Solution by JOHN DOLMAN, Jr., Philadelphia, Pennsylvania.

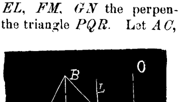
If the random point be taken upon the circumference of a variable circle, concentric with the given circle, the area of the circular "locus" is evidently always one quarter of this variable circle. Again: As the random points are equably distributed over the area of the given circle, the variable circle must always increase in area by equal increments, and, as it varies from zero to the full area of the given circle, its mean area is, of course, half of the given circle. Therefore the mean area of the locus named is one half of one quarter or one eighth of the given circle.

23. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Find the average area of all the triangles that can be drawn *perpendicular-sided* to a given plane scalene triangle.

Solution by G. B. M. ZERR, A. M., Ph. D., Vice-President and Professor of Mathematics and Sciences, Inter State College, Texarkana, Texas.

Let ABC, be the given scalene triangle; EL, FM, GN the perpendiculars to AC, AB, BC respectively, forming the triangle PQR. Let AC, AO be the axes of co-ordinates. Draw BD, FH, GK perpendicular to AC.



Let $AD=d$, $BC=e$, $BD=h$, $AC=b$, $AB=c$, $BC=a$, $AE=u$, $AH=v$, $AK=w$.

Then $FH=hv/d$, $GK=\dfrac{h(b-w)}{e}$. $x=u$,

$$y=\frac{hv}{d}+\frac{dv}{h}-\frac{dx}{h},\quad y=\frac{h(b-w)}{e}+\frac{(b-d)(x-w)}{h},$$

are the equations to LE, FM, and GN respectively.

$$\therefore\ \frac{h^2ev+d^2ev+w(h^2d-d^2e+bde)-h^2db}{bde}=\frac{c^2ev+a^2dw-h^2db}{bde},\ \text{is the}$$

abscissa of Q. Let A_1=area PQR, $\triangle$= area ABC.

$$\therefore\ A_1=\frac{\triangle\,(c^2ev+a^2dw-h^2db-bdeu)^2}{b^2d^2e^2h^2}=\frac{1}{2}\,\frac{(c^2ev+a^2dw-h^2db-bdeu)^2}{bd^2e^2h}.$$

The limits of u are 0 and b; of v, 0 and d; of w, d and b.

$$\therefore\ A=\frac{\int_0^b\int_0^d\int_d^b A_1\,du\,dv\,dw}{\int_0^b\int_0^d\int_d^b du\,dv\,dw}=\frac{1}{bde}\int_0^b\int_0^d\int_d^b A_1\,du\,dv\,dw$$

$$=\frac{1}{6a^2b^2d^4e^3h}\int_0^b\int_0^d\left\{(c^2ev+e^2db-bdeu)^3-(c^2ev-h^2bd+a^2d^2-bdeu)^3\right\}du\,dv$$

$$=\frac{1}{24a^2b^2c^2e^4h}\int_0^b\left\{(c^2e+e^2b-beu)^4-(e^2b-beu)^4-(c^2e-h^2b+a^2d-beu)^4\right.$$

$$+(a^2d-h^2b-beu)^4 \Big\} du$$

$$=\frac{1}{60a^2b^3c^2h}\Big\{ (a^2+bd)^5-(a^2-be)^5-b^5e^5-b^5d^5 \Big\}$$

$$=\frac{1}{120\triangle a^2b^2c^2}\left\{\left(\frac{a^2+b^2+c^2}{2}\right)^5-\left(\frac{a^2+c^2-b^2}{2}\right)^5-\left(\frac{a^2+b^2-c^2}{2}\right)^5 -\left(\frac{b^2+c^2-a^2}{2}\right)^5\right\}$$

$$=\frac{4\triangle^4}{15a^2b^2c^2}\Big\{ (\cot A+\cot B+\cot C)^5-\cot^5 A-\cot^5 B-\cot^5 C \Big\}$$

$$=\frac{a^4+b^4+c^4}{48\triangle}.$$

The last four expressions are the same and show the beautiful relations existing between the terms of the triangle.

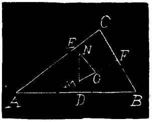
This problem was also solved by *F. P. MATZ*. His solution is published in this issue as solution of problem 24, that problem being identical with 23.

24. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

The average area of the triangle formed by three perpendiculars drawn from the sides of the triangle (a, b, c), is $\mathbf{A}=(a^4+b^4+c^4)/48\triangle$.

Solution by the PROPOSER.

Let $BC=a$, $CA=b$, $AB=c$, $BF=x$, $CE=y$, and $AD=z$. The $\triangle MNO$ is similar to the $\triangle ABC$; and it may be *wholly within*, *partly within*, or *wholly without*, the $\triangle ABC$. Put $\angle NAD=\phi$, and $\angle MBD=\theta$; then $AN=(b-y)/\cos(A-\phi)$ $=z\cos\phi \ldots\therefore(\alpha)$, and $ND=z\tan\phi\ldots.(\beta)$. By means of (α), we have from (β), $ND=[(b-y)-z\cos A]/\sin A\ldots.(1)$. Also, $BM=(c-z)/\cos(B-\theta)=x/\cos\theta\ldots.(\gamma)$, and $MD=(c-z)\tan(B-\theta)\ldots\therefore(\delta)$. By means of (γ) we have from (δ), $MD=[x-(c-z)\cos B]/\sin B\ldots.(2)$. Subtracting (2) from (1), we have the expression:



$$MN=\frac{b\sin B+c\sin A\cos B-(x\sin A+y\sin B+z\sin C)}{\sin A\sin B}\ldots.(3).$$

The *first two* terms in the numerator of (3) may be symmetrically written thus: $a\sin B\cos C+b\sin C\cos A+c\sin A\cos B,=\frac{1}{2}(a\sin A+b\sin B+c\sin C)\ldots.(\varepsilon)$. Transforming (3) by means of (ε), we have

$$MN=\frac{(a-2x)\sin A+(b-2y)\sin B+(c-2z)\sin C}{2\sin A\sin B}\ldots.(4).$$

Representing the numerator of the right-hand member of (4) by $\mathbf{N}$, we have (by symmetry) the expressions:

$NO=\mathbf{N}/2\sin C\sin A$, and $OM=\mathbf{N}/2\sin B\sin C$;

and, consequently, the $\triangle MNO=\frac{1}{8}(\mathbf{N})^2 \,/\, \sin A \sin B \sin C \ldots (5)$. The expression for the average area of the $\triangle MNO$, therefore, becomes

$$\mathbf{A}=\frac{1}{8abc\sin A\sin B\sin C}\int_o^a\int_o^b\int_o^c(\mathbf{N})^2 dxdydz$$

$$=\tfrac{1}{24}\left(\frac{a^2\sin A}{\sin B\sin C}+\frac{b^2\sin B}{\sin C\sin A}+\frac{c^2\sin C}{\sin A\sin B}\right)=\frac{a^4+b^4+c^4}{48\triangle}.$$

[Note.—Problems twenty-three and twenty-four are identical. This fact was not observed until after they were both printed. ED.]

PROBLEMS.

31. **Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.**

Find the average length of a line drawn at random across the opposite sides of a rectangle whose length is l and breadth b.

32. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.**

Find the average area of the random sector whose vertex is a random point in a given circle.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

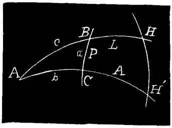
SOLUTIONS OF PROBLEMS.

17. **Proposed by SAMUEL HART WRIGHT, M. D., M. A., Ph. D. Penn Yan, New York.**

A bright star passed my meridian at 7 P. M. The Chronometer soon after ran down and stopped, but I set it again when the same star had a true altitude of $30^\circ=\alpha$. What time was it then, my latitude being $42^\circ\ 30'$ N.$=\lambda$, and the star's declination 60° N.?$=\delta$

II. Solution by the PROPOSER.

Let B be the north pole, A the zenith, C the star, HH' the horizon, AH and AH' each $=90^\circ$, AH' being a meridian, AH a verticle circle, BH' the altitude of the pole = the latitude $=L$, AB=co-latitude $=c$, $BC=a=$ polar distance of the star$=P$, $AC=b=$the zenith distance of star, $CH=A=$altitude of the star, and the angle $ABC=$the hour-angle of the star$=T$ in siderial time. Put $s=\frac{1}{2}(a+b+c)$, and $s-a=a'$, $s-b=b'$, and $s-c=c'$. Then by *Sph. Trig.*



$\sin\frac{1}{2}T=\sqrt{\left(\frac{\sin c'\sin a'}{\sin c\sin a}\right)}$, and

$\frac{1}{2}T=51^\circ 40' 18''.5 \ldots (1)$ or $\cos\frac{1}{2}T=\sqrt{[\sin s \sin (s-b) \operatorname{cosec} c \operatorname{cosec} a]}$, and $\frac{1}{2}T=51^\circ 40' 18''.5 \ldots (2)$, or in terms of A, L, and P; put $s'=\frac{1}{2}(A+L+P)$, then $s=\frac{1}{2}(180^\circ-A-L+P)=90^\circ-s'+P=90^\circ-A-L+s'$, and $s-P=90^\circ-s'$, and $s-(90^\circ-L)=s'-A$. Whence, $\sin\frac{1}{2}T=\sqrt{[\sec L \operatorname{cosec} P \cos s' \sin (s'-A)]}$, and $\frac{1}{2}T=51^\circ 40' 18''.5 \ldots (3)$.

$\therefore$ $T=103^\circ 20' 37''=6$ hr. 53 min. 22.467 sec. of siderial time=6 hr. 52 min. 14.75 sec. mean solar time. To this add 7 hrs., the time the star was on the meridian, and we get 1 hr. 52 min. 14.75 sec. of the morning of the next day, for the time when the chronometer was set.

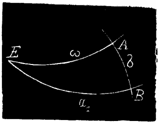
In Bowditch's *Practical Navigator*, pp. 209-210, the rules for finding T are translations of eqs. (2) and (3), but no reasons for the rules are given, and no formulas from which they are derived. The above formulas, (1), (2), and (3) are as applicable for obtaining correct time on land as at sea. [This solution is important as showing how the Rule in Bowditch's *Navigator* is obtained,—which some very good mathematicians have failed to comprehend.—EDITOR.]

20. Proposed by SAMUEL HART WRIGHT, M. D., M. A., Ph. D., Penn Yan, New York.

When does the Dog-Star and the Sun rise together in latitude $42^\circ 30'$ N.$=\lambda$, given the R. A. of Sirius$=6$ hrs. 40 min. 30 sec., and its Dec.$=16^\circ 33' 56''$ S.?

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science in Inter State College, Texarkana, Texas.

Let $\lambda=$ latitude of observer, $\alpha=$R. A., $\delta=$declination, $t=$hour angle of Sirius. $\alpha_1=$R. A., $\delta_1=$declination, $t_1=$hour angle of sun, $\varepsilon=$obliquity of the ecliptic, $\omega=$ distance from vernal equinox to the sun's position, $\tau=$ time of sun-rise before six o'clock. Then we get $\cos t=-\tan\lambda\tan\delta \ldots (1)$. $\cos t=-\tan\lambda\tan\delta_1 \ldots (2)$. $\sin\alpha_1=\tan\delta_1 \cot\varepsilon \ldots (3)$. $\alpha_1-t_1=\alpha-t=\theta$, or $\alpha_1=\theta+t_1 \ldots (4)$. $\sin\alpha_1=\sin(\theta+t_1)\ldots(5)$. $\cos\varepsilon=\cot\omega \tan\alpha_4 \ldots (6)$. $\sin\tau=\tan\lambda\tan\delta_1 \ldots (7)$. From



(3) and (5), $\sin(\theta+t_1)=\tan\delta_1\cot\varepsilon \ldots (8)$. From (2) and (8),

$$\tan\delta_1=\frac{\sin(\theta+t_1)}{\cot\varepsilon}=-\frac{\cos t_1}{\tan\lambda}\ldots(9).$$

From (9), $\tan t_1=-\dfrac{\cot\varepsilon+\sin\theta\tan\lambda}{\cos\theta\tan\lambda}\ldots(10)$.

But $\lambda=42^\circ 30'$, $\alpha=6$ hr. 40 min. 30 sec., $\delta=16^\circ 33' 56''$ S., $\varepsilon=23^\circ 27' 13''$.

From (1), $t=74^\circ 10' 57''.81=4$ h. 56 m. 43.85 sec.

$\alpha-t=\theta=1$ h. 43 m. 46.15 sec. $=25^\circ 56' 32''.25=$siderial time when Sirius rises.

From (10), $t_1=107^\circ 3' 20''=7$h. 8 m. 13.33 sec.

From (4), $\alpha_1=t_1+\theta=8$ h. 51 m. 59.48 sec.$=132^\circ 59' 52''.25$.

From (3), $\delta_1=17^\circ 36' 16''.85$. From (6), $\omega=130^\circ 32' 38''.5$.

From (7), $\tau=16^\circ\ 54'\ 12''.41=1$ h. 7 m. 36.83 sec. before six o'clock.

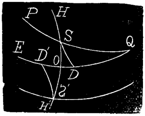
$\therefore$ The sun rises 4 h. 52 m. 23.17 sec. after mid-night. It takes the sun 186 days to go from the vernal to the autumnal equinoxes.

$\therefore\ 180^\circ : 186=130^\circ\ 32'\ 38''.5 : 134.8955$.

March $20+134.8955=$ August 2. $\therefore$ The event takes place August 2, 4 h. 52 m. 23.17 sec. To find the time when the sun and Sirius set together we have $\theta=a+t=a_1+t_1=11$ h. 37 m. 13.85 sec. The rest of the calculation is the same as that given above.

III. Solution by the PROPOSER.

Let HH' be the horizon, EQ the equator, PQ the ecliptic, S the sun at rising, S' Sirius at rising, $D'S'$ the Dec. $=\delta$, $D'Q=12$ h. $-$ R. A. $=79^\circ$ $52'\ 30''=\alpha'$, $PQE=$ apparent obliquity of the ecliptic $=23^\circ\ 27'\ 20''=B$, $S'OD'=SOD=90^\circ-\lambda$, $SQ=a$ $=180^\circ$--sun's Long. Required DS the sun's Dec. north. Then $\sin D'O=\tan\lambda\tan\delta$, and $DO=15^\circ\ 49'$ $2''=m$, and $OQ=\alpha-m=64^\circ\ 3'\ 28''=c$, and



$$\cot a=\frac{\cos(B-\gamma')\cot c}{\cos\gamma'},$$

and $a=49^\circ\ 34'\ 22''$, where $\cot\gamma'=\cot\lambda\cos c$, and $\gamma'=64^\circ\ 28'\ 47''$. Whence $\sin DS=\sin B\sin a$. $\therefore DS=17^\circ\ 38'\ 16''$ north, and sun's Long. $180^\circ-a=130^\circ\ 25'\ 38''$. The sun's Dec. and Long. found, give by the common solar Tables, the date of August 2d, eleven days before and after this event is the beginning and end of "Dog-Days."

It is well to note that the assertion of Prof. Matz, in his solution in the June number, that the hour-angles of the Dog-Star and the sun are equal, is not true. For the hour-angle of the sun must be the supplement of that of the Dog-Star, instead of equal to it. That of course makes the hour-angle of the Dog-Star from the upper meridian=that of the sun from the lower meridian, as it should have been stated.

NOTE.—On page 134, Prof. Matz asserts that when the moon is $\frac{2}{3}$ through her last quarter she will have a reversed crescent of the same size as when $\frac{2}{3}$ through her *first* quarter. This is not true. The Moon can be only $\frac{1}{3}$ through her last quarter, when such a crescent is seen.—S. H. W.

PROBLEMS.

31. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

In order that a vertical cylindric stalk may be severed by a blow of minimum force, the stalk must be struck at what inclination by a sharp wedge-shaped blade?

32. Proposed by S. H. WRIGHT, M. D., M. A., Ph. D., Penn Yan, New York.

Intermittent reflections of flashes of light on a clear sky after dark, indicated a storm was progressing *below* the horizon. Refraction of 34′ on the horizon, brought the upper edge of the storm-cloud up to the horizon, and was just visible. How far off was the storm if the cloud was one mile above the earth?

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

The American Mathematical Monthly in Spain.

The leading article of the Spanish Journal "El Progreso Mathematico" for March 1895 speaks of "la rica literatura que forman en conjunto las abras escritas en estos ultimos anos acerca de los diversos sistemas geometricos de los Sres. Flye Ste. Marie, Frischauf, Erdmann, Lipschitz, Scheffler, Killing, Battaglini, Cayley, Klein, Poincare, Bruce Halsted(* * *), Vassilief;" and the note is "(* * *) Este ilustrado profesor de la Universidad de Texas publica actualmente una serie de articulos sobre la Geometria non-Euclidea en the *American Mathematical Monthly*."

The trisection problem is considered in Petersen's Methods and Theories (London 1879) p. 101. Also in the recently published Vortrage uber ansgewablte Fragen der Elementargeomentrie, von. F. Klein, ausgearbeitet von F. Torgert (Leipzig 1895), p. 11.

DAVID E. SMITH.

I reply to W. E. Heals inquiry on page 171 of AMERACAN MATHEMATICAL MONTHLY as to the impossibility of trisecting an angle with rule and compass as follows:

We can draw only circles and straight lines with a compass and rule and can therefore express by that means only the roots of quadratics. By means of the compass we can find the square root. But the equation $\sin 3A=3\sin A-4\sin^3 A$ is a cubic and we have no method of finding the cube root by means of a compass.

OTTO CLAYTON.

NOTES ON THE DEMONSTRATION OF EUCLID'S ELEVENTH AXIOM.

NOTE 1.—To complete my demonstration in the May No. GF (in *third*) should be solved equal to the given *constant* DA. If GF be supposed greater than DA, then GD and FA would converge toward DA, and when folded over on axis DA, the corresponding lines DM and AN would diverge, making GDM and FAN broken lines, whereas, by hypothesis, they are straight lines. Similarly, if GF be supposed less than DA.

NOTE 2.—Since mathematicians differ about the definition of parallels, might not the term *parallel* be dispensed with and *equidistant* be substituted?

WARREN HOLDEN,
Girard College, Philadelphia.

ANSWER TO "READER'S" QUERY.

Some one signing himself "Reader" in the February issue, 1895, makes inquiry as to the meaning of the word "psendo spherical". He will find an answer to that part of his question in my article in the May No. of the MONTHLY. *Numerous* points of "difference" between the Euclidian and non-Euclidian Geometry are also indicated.

"Reader" will find the term "Hyper-space" in the article on "Measurment" by R. S. Bell in the Encyclopaedia Brittanica, Ninth Edition. I do not find the expression "Ideal Space" in either of my articles in the November issue of last year. Kant maintained that space is purely subjective. Fichte reasoned that if space is subjective, the material bodies contained in it must also be subjective. This is Idealism. The Kantian view of space may appropriately be called Idealistic inasmuch as it is avowedly purely subjective.

JOHN N. LYLE.

EDITORIALS.

THE MONTHLY will be published at Kidder until January after which it will probably be published at Springfield.

DR. F. P. MATZ has accepted the Professorship of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania. This is one of the most flourishing Colleges in Pennsylvania; and Dr. Matz is the recipient of a handsome salary, as well as the holder of a very desirable position.

DR. ALEXANDER MACFARLANE, a contributor to the MONTHLY, and a distinguished Physicist and Electrician, formerly Professor of Physics in the University of Texas, has accepted the chair of Lecturer in Electrical Engineering in Lehigh University. Dr. Macfarlane's reputation as an investigator in Mathematical Physics is international and the University to which he has been called is to be congratulated for selecting such an able man.

DR. E. S. LOOMIS, for the last ten years professor of Mathematics and principal of the Normal Department in Baldwin University, has been elected Professor of mathematics in the West High School of Cleveland. This appointment, coming as it does, at this time, from our neighboring city of Cleveland, cannot be considered otherwise than as a vindication of Dr. Loomis' eminent ability as a teacher of mathematics, and also shows that notwithstanding his doubts that Jonah "swallowed the whale," the Cleveland Educational authorities believe him still capable of teaching that science.—*Bered Advertiser.*

We congratulate the Doctor upon his good fortune in being called to a position in which his superior ability will be fully appreciated.

THE National Normal University, Lebanon, Ohio, of whicn Dr. Alfred Holbrook is the President, recently conferred (with the highest distinction) the Degree of *Doctor of Science* (Sc. D.), upon our valued contributor, F. P. Matz. The subjects of the theses submitted by him, were: (1) *The Conditions*

Necessary to the Existence of Life and Mind; (2) *Neo-Vitalism considered from a Monistic Standpoint;* (3) *New Methods for the Derivation of the Formulae for the Perturbations of Comets.*

DR. G. A. MILLER, Professor of Mathematics in the University of Michigan, has gone to Leipzig, Germany, to pursue his study in Mathematics. Dr. Miller is contributing, to the MONTHLY, a series of articles on *Substitution Groups* for which we are very thankful. This difficult subject is receiving a great deal of attention from eminent Mathematicians. Dr. Miller is now presenting the subject to the readers of the MONTHLY in a way that will be greatly appreciated.

DRS. ZERR and Matz both sent *strong* replies to Counselor Dolman's Comments on Problems 14 and 15, Average and Probability, but we thought best not to use any more space in the discussion. The published solution of problem 14 is unquestionably correct.

PROFS. J. F. W. SCHEFFER and A. H. Bell should have received credit for solutions of problem 49, Arithmetic Department.

WE HAVE received several calls for copies of the following books: *Salmon's Treatise on the Higher Plane Curves* (Latest Edition); *Salmon's Treatise on the Analytic Geometry of Three Dimensions* (4th Edn. 1882); *Salmon's Modern Higher Algebra; B. Price's Treatise on the Infinitesimal Calculus* (4 vol.). If any of our readers have copies of any of the above named books and wish to dispose of the same they should write to us at Springfield, Mo. Or, if any of our readers have other books and wish to dispose of them, we shall be pleased to publish the name of such books and the price of same, in the next issue of the MONTHLY. We can thus be of service to those wishing to obtain books and those wishing to dispose of them.

ALL communications and subscriptions should be sent to B. F Finkel, Springfield, Mo. Persons failing to receive their copies should write to the Publishers, Kidder, Mo.

BOOKS AND PERIODICALS.

Algebra for Beginners. By H. S. Hall and S. R. Knight. Revised and Adopted to American Schools by Frank L. Sevenoak, A. M., M. D., Professor of Mathematics and Assistant Principal in Stevens School, Academic Department of Stevens Institute of Technology.

Small 8vo. cloth, VIII+188 pp. Price, 60 cents. New York: Macmillan & Co.

This neat little work treats of all subjects to and including quadratic euqations usually presented in an elementary algebra. The aim of the reviser has been to thoroughly adopt the book to the wants of all those who do not require a knowledge of Algebra beyond Quadratic Equations. In this, he has not failed. Each subject contains a long list of well selected examples, a very commendable feature in any Mathematical text-book. B. F. F

Bellum Helveticum for Beginners in Latin. By Cornelius Marshal Lowe, Ph. D., Heidelberg University, and Nathaniel Butler, Jr., M. A., University of Chicago. 8vo. cloth, 312 pp. Chicago: Albert, Scott & Co.

This book is an introduction to the reading of Latin Authors, based on the inductive method illustrating the forms and construction of classical Latin prose. The book is written to satisfy the demand of many teachers that Latin be made at the beginning, a living subject for the student. We consider the book a most excellent one and do not hesitate to recommend it to the favorable consideration of teachers of Latin. B. F. F.

The Essentials of Arithmetic, Oral and Written. By Gordon A. Southworth, Superintendent of Schools, Somerville, Massachusetts. 8vo. cloth, 186 pp. Boston, New York and Chicago: Leach, Shewell & Sandborne.

This book is designed for use in the third, fourth, fifth, and sixth-year grades of Public Schools. Oral and written exercises are happily blended throughout the entire work. The answers to the exercises are appended at the close of the book. B. F. F.

An Elementry Algebra, Theoretical and Practical. By J. W. Nicholson, A. M., President and Professor of Mathematics in the Louisana State University and Agricultural and Mechanical College. 8vo. Half Leather back, 284 pp. New York and New Orleans: University Publishing Co.

This is a splendid elementary algebra in which the author has treated a number of subjects in an original manner. The book contains about 2500 problems.

B. F. F.

Elements of Geometry, after Legendre with a Selection of Geometrical Exercises, and Hints for the Solution of the same. By Charles S. Venable LL. D., Professor of Mathematics in the University of Virginia. 8vo. cloth, 413 pp. New York and New Orleans: University Publishing Co.

This is a complete translation and adaptation of the latest edition of the standard work of Legendre. A number of changes have been made. These consist mainly in the discussion of parallels; in the treatment of tangencies; in the addition of some theorems and the omission of a few; the substitution of the method of limits for the *reducto ad absurdum* in the treatment of the measure of the circle and of the "three round bodies;" etc. All these changes are for the best. Much might be said in commendation of this excellent work. B. F. F.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Albert Shaw. Price, $2.50 per year. Single Number, 25 cents. The Review of Reviews Co., New York City.

The leading features of the August *Review of Reviews* are: "Theodore Roosevelt," a character sketch by Julian Ralph; "The Clearing of Mulberry Bend," the story of the rise and fall of a New York slum, by Jacob A. Riis; "The Third Salisbury Cabinet," by W. T. Stead, and "The Record of the Rosebery Administration,"—all four articles well illustrated. The *Review of Reviews* is an illustrated summary of the world's progress. B. F. F.

The Cosmopolitan: An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year. Single Number, 10 cents.

ERRATUM.

In Dr. Martin's solution of Diophantine Problem 15, p. 439, Vol. I.
for "$(4m+3)^2=16m^2+24m+1=8(2m^2+3\text{m})+1$"
read $4(m+3)^2=16m^2+24m+9=8(2m^2+3m+1)+1$.